INTEGRATOR BLOCK OFF – GRID POINTS COLLOCATION METHOD FOR DIRECT SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS USING CHEBYSHEV POLYNOMIALS AS BASIS FUNCTION.

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ABSTRACT: *The numerical computation of differential equations cannot be overemphasized as it is evident in the literatures. It has been observed that analytical solution of some differential equations are intractable, hence there is need to seek for an alternative solution to such equations. Circumventing this problem resulted into an approximate solution otherwise known as numerical solution. There are so many numerical methods that can be used in solving differential equations which include predictor – corrector method which is linear multistep in nature and not self-starting method. In this presentation the focus is on presenting a self-starting multistep method for direct solution of Second Order Ordinary Differential Equations as against the popular predictor – corrector method which needs additional value for starting point which may alter the accuracy of the method. The method is a mixture of grid and off grid collocation point and often refer to as Block linear multistep method.*

KEYWORDS: Collocation, Interpolation, Predictor – Corrector, Chebyshev Polynomials, Integrator Off –Grid, Block Method.

INTRODUCTION

In seeking for an approximate solution to differential equations, the role of numerical computation occupies a central point. It is an understatement to say that not all differential equations have analytical solution hence calls for an approximate solution to such differential equations. In the quest for seeking for an approximate method in solving differential equations, so many authors have worked extensively in order to come up with various approximate methods. The popular and highly celebrated among these methods is the Linear Multistep method which is predictor – corrector method.

Many authors have used various approaches in deriving numerical integrators in handling Ordinary Differential Equations which include among others [5, 7, 8, 10, 11, 14] to mention just a view. They all used the method of power series as basis function in deriving various multistep methods in solving Ordinary Differential Equations which are predictor – corrector in nature. One of the major set back of this method is that it requires an additional information (value) as a staring value and if nor appropriately chosen, it may alter the accuracy of the solution, more so it is often laborious in deriving an appropriate corrector for a given predictor.

In order to surmount the problems of predictor – corrector method came the introduction of Block Linear Multistep Method. The method is self-starting which requires no additional value to start the computation thereby reducing the errors as the computation progresses. This particular method has been used by many authors and found to be efficient. Some of the authors include [4, 6, 9, 12] among others.

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In this write –up, A Block off Grid Collocation Point Linear Multistep Method for direct solution of Second Order Ordinary Differential Equations shall be presented using Chebyshev polynomial as basis function. Chebyshev polynomial is chosen because of its high accuracy level among other monomials [8]. This approach has been used by many authors such as [1, 2, 3]

The Method

Chebyshev polynomial which can be define as [3]

$$y(x) = \sum_{r=0}^{\infty} a_r T_r(x) \qquad (1)$$

that is

$$y(x) = \sum_{r=0}^{\infty} a_r T_r \left(\frac{2x-2kh-nh}{nh}\right) \qquad (2)$$

In which

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \qquad (3)$$

And that

$$T_0(x) = 1\,, \quad T_1(x) = x \qquad (4)$$

Now letting

$$y(x) = \sum_{r=0}^{\infty} a_r T_r(x) \qquad (5)$$

Equation (3) can be used recursively to generate the Chebyshev polynomial. In this presentation, a Second Order Ordinary Differential Equation is considered in which collocation is done in the closed interval $[x_n \;, x_{n+2}]$ with step size of $\frac{1}{3}$, while interpolation is done at two grid points, that is x_n , x_{n+1} . From the foregoing, the polynomial equation considered in using equation (5) and according to [8] yields

$$y(x) = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4(8x^4 - 8x^2 + 1) + a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 48x^4 + 18x^2 - 1) + a_7(64x^7 - 112x^5 + 56x^3 - 7x) + a_8(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) \qquad (6)$$

Differentiating equation (6) twice leads to

$$y''(x) = 4a_2 + 24xa_3 + (96x^2 - 16)a_4 + (320x^3 - 120x)a_5 + (960x^4 - 576x^2 + 36)a_6 + (2688x^5 - 2240x^3 + 336x)\, a_7 + a_8(7168x^6 - 7680x^4 + 1920x^2 - 64) \qquad (7)$$

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Interpolating equation (6) at $= x_n$ and x_{n+1} , and collocating equation (7) at $x = x_{n+r}, r = 0\left(\frac{1}{3}\right)2$, yields the matrix equation

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & -24 & 80 & -200 & 420 & -784 & 1344 \\ 0 & 0 & 2916 & -11664 & 19440 & -10800 & -22140 & 62496 & -71744 \\ 0 & 0 & 2916 & -5832 & -3888 & 20520 & -11772 & -29232 & 46912 \\ 0 & 0 & -16 & 0 & -16 & 0 & 36 & 0 & -64 \\ 0 & 0 & 2916 & 5832 & -3888 & -20520 & -11772 & 29232 & 46912 \\ 0 & 0 & 2916 & 11664 & 19440 & 10800 & -22140 & -62496 & -71744 \\ 0 & 0 & 4 & -24 & 80 & -200 & 420 & -784 & 1344 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{3}} \\ h^2 f_{n+\frac{2}{3}} \\ h^2 f_{n+1} \\ h^2 f_{n+\frac{4}{3}} \\ h^2 f_{n+\frac{5}{3}} \\ h^2 f_{n+2} \end{pmatrix}$$

Solving the system of matrix equations above leads to the following:

$$a_0 = y_{n+1} + \frac{h^2}{81920}\left(53f_n + 7900f_{n+1} + 53f_{n+2} + 1602f_{n+\frac{1}{3}} + 4635f_{n+\frac{2}{3}} + 4635f_{n+\frac{4}{3}} + 1602f_{n+\frac{5}{3}}\right)$$

$$a_1 = y_{n+1} - y_n + \frac{h^2}{215040}\left(1281f_n + 36032f_{n+1} + 223f_{n+2} + 17676f_{n+\frac{1}{3}} + 24453f_{n+\frac{2}{3}} + 19611f_{n+\frac{4}{3}} + 8244f_{n+\frac{5}{3}}\right)$$

$$a_2 = \frac{h^2}{30720}\left(37f_n + 2260f_{n+1} + 37f_{n+2} + 918f_{n+\frac{1}{3}} + 1755f_{n+\frac{2}{3}} + 1755f_{n+\frac{4}{3}} + 918f_{n+\frac{5}{3}}\right)$$

$$a_3 = \frac{h^2}{30720}\left(-37f_n - 37f_{n+2} - 612f_{n+\frac{1}{3}} - 585f_{n+\frac{2}{3}} + 585f_{n+\frac{4}{3}} + 612f_{n+\frac{5}{3}}\right)$$

$$a_4 = \frac{h^2}{61440}\left(59f_n - 604f_{n+1} + 59f_{n+2} + 702f_{n+\frac{1}{3}} - 459f_{n+\frac{2}{3}} - 459f_{n+\frac{4}{3}} + 702f_{n+\frac{5}{3}}\right)$$

$$a_5 = \frac{h^2}{10240}\left(-9f_n + 9f_{n+2} - 36f_{n+\frac{1}{3}} + 99f_{n+\frac{2}{3}} - 99f_{n+\frac{4}{3}} + 36f_{n+\frac{5}{3}}\right)$$

$$a_6 = \frac{h^2}{71680}\left(39f_n + 732f_{n+1} + 39f_{n+2} + 18f_{n+\frac{1}{3}} - 423f_{n+\frac{2}{3}} - 423f_{n+\frac{4}{3}} + 18f_{n+\frac{5}{3}}\right)$$

$$a_7 = \frac{h^2}{71680}\left(-27f_n + 27f_{n+2} + 108f_{n+\frac{1}{3}} - 135f_{n+\frac{2}{3}} + 135f_{n+\frac{4}{3}} - 108f_{n+\frac{5}{3}}\right)$$

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$$a_8 = \frac{h^2}{573440}\left(81f_n - 1620f_{n+1} + 81f_{n+2} - 486f_{n+\frac{1}{3}} + 1215f_{n+\frac{2}{3}} + 1215f_{n+\frac{4}{3}} - 486f_{n+\frac{5}{3}}\right)$$

Substituting these values into equation (6) and by letting

$$x = \frac{x-kh-h}{h} = t$$

yields the continuous formula

$$y_{(x)} = y_{n+1}(t+1) - y_n t +$$

$$\frac{h^2 f_n}{1720320}(13440t - 14336t^3 + 7168t^4 + 48384t^5 - 32256t^6 - 41472t^7 + 31104t^8) +$$

$$\frac{h^2 f_{n+\frac{1}{3}}}{1720320}(195840t + 129024t^3 - 96768t^4 - 387072t^5 + 387072t^6 + 165888t^7 - 186624t^8) +$$

$$\frac{h^2 f_{n+\frac{2}{3}}}{1720320}(399744t - 645120t^3 + 967680t^4 + 628992t^5 - 1257984t^6 - 207360t^7 + 466560t^8)$$

$$+ \frac{h^2 f_{n+1}}{1720320}(288256t + 860160t^2 - 1756160t^4 + 1806336t^6 - 622080t^8) +$$

$$+ \frac{h^2 f_{n+\frac{4}{3}}}{1720320}(-47232t + 645120t^3 + 967680t^4 - 628992t^5 - 1257984t^6 + 207360t^7 + 466560t^8) +$$

$$+ \frac{h^2 f_{n+\frac{5}{3}}}{1720320}(11520t - 129024t^3 - 96768t^4 + 387072t^5 + 387072t^6 - 165888t^7 - 186624t^8) +$$

$$\frac{h^2 f_{n+2}}{1720320}(-1408t + 14336t^3 + 7168t^4 - 48384t^5 - 32256t^6 + 41472t^7 + 31104t^8)$$

Evaluating the above equation at the points $t = 1\,(\frac{1}{3})2$ and letting $y_{(x)} = y_{n+2}$ leads to

$$\alpha y_{n+2} = \beta y_{n+1} - \gamma y_n + \varphi f_i \left(i = 1\left(\frac{1}{3}\right)2\right)$$

Where

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{8}{3} \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \end{pmatrix} \text{ and}$$

$$\varphi = \begin{pmatrix} \frac{47}{6720} & \frac{27}{224} & \frac{459}{2240} & \frac{563}{1680} & \frac{459}{2240} & \frac{27}{224} & \frac{47}{6720} \\ \frac{2819}{2240} & \frac{-9297}{1120} & \frac{56817}{2240} & \frac{-22047}{560} & \frac{85797}{2240} & \frac{-22401}{1120} & \frac{12871}{2240} \\ \frac{661}{272160} & \frac{1789}{45360} & \frac{2147}{30240} & \frac{6817}{68040} & \frac{841}{90720} & -\frac{1}{15120} & -\frac{11}{272160} \\ \frac{535}{108864} & \frac{475}{6048} & \frac{5167}{36288} & \frac{5725}{27216} & \frac{1321}{12096} & \frac{193}{18144} & \frac{-53}{108864} \\ \frac{4627}{272160} & \frac{691}{6480} & \frac{629}{1440} & \frac{1687}{9720} & \frac{7681}{12960} & \frac{43}{720} & \frac{6577}{38880} \\ \frac{4945}{27216} & \frac{-55}{56} & \frac{35029}{9072} & \frac{-139597}{27216} & \frac{6157}{1008} & \frac{-13415}{4536} & \frac{33841}{27216} \end{pmatrix}$$

Order and Error Constant

Definition: According to [13], the order and error constant of Linear multistep method can be defined as "The Linear operator and its associated block formula is said to be of order$p$, if $C_0 = C_1 = \ . \ . \ . = \ C_{p+1} = 0 \ and \ C_{p+2} \neq \ 0.$ C_{p+2} is said to the error constant or the truncation error and the error constant is $C_{p+k} = \ C_{p+2}h^{p+2}y^{p+2}(x) + 0(h^{p+3})$

In calculating the error constant for the method, it was discovered that the method is of order 7 with error constant of

$$C_9 = \left(\frac{1045}{17658017412} \quad \frac{224}{16851671381} \quad \frac{7}{2431512} \quad \frac{35}{11068107} \quad \frac{1581}{168471289} \quad \frac{243}{116451167}\right)^T$$

Numerical Experiments

At this point, some numerical illustrations were considered in which comparison shall be made between the analytical and approximate solution with step length of $h = \ 0.1$

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Illustration I: Solve the Cauchy – Euler differential equation

$$x^2y'' + 1.5xy' - 0.5y = 0: \quad given\ that\ \ y(1) = 2, and\ y'(1) = 5\ ,$$

in the interval of [1, 2] with $h\ = 0.1$. The analytic solution is

$$y(x) = 4x^{0.5} + 2x^{-1}.$$

The table below shows the numerical solution to illustration I in which the exact and numerical solutions were compared.

TABLE I COMPUTATIONAL RESULT TO ILLUSTRATION I

X	EXACT SOLUTION	Y - COMPUTED	ERROR
1.0	6.0000000000	5.981613212	0.018386788
1.1	6.013417211	6.001267562	0.012149649
1.2	6.048447127	6.048442162	4.965 E - 06
1.3	6.099163239	6.099161214	2.025 E- 06
1.4	6.161435255	6.161433123	2.132 E- 06
1.5	6.232312819	6.232312609	2.1 E – 07
1.6	6.309644256	6.309642112	2.114 E -06
1.7	6.391832512	6.391830428	2.084 E -06
1.8	6.477674257	6.477671193	3.004 E -06
1.9	6.5662511080	6.566250471	6.09 E -07
2.0	6.656854249	6.656853115	1.134 E -06

Illustration II Find the general Solution to the differential equation:

$$x^2y'' - 5xy' + 9y = 0: \quad in\ which\ y(1) = 2, and\ y'(1) = 3\ ,$$

in the interval of [0, 1] with $h\ = 0.1$. The analytical solution is

$$y(x) = \ x^3(2 - 3\ln x).$$

Table II below shows the comparison between the computed result and the analytical solution. The solution is presented to 10 significant figures.

TABLE II: NUMERICAL COMPUTATION TO ILLUSTRATION II

X	Y- EXACT	Y - COMPUTED	ERROR
0	0.0000000000	0.0000000000	0.0000000000
0.1	0.008907755279	0.00897754279	9.99982E -10
0.2	0.006828313737	0.006828302964	1.0773E – 8
0.3	0.04489534652	0.04489530716	3.936 E – 8
0.4	0.3039278213	0.3039277634	5.79 E – 8
0.5	0.509901916	0.5099300724	1.192 E – 7
0.6	0.7630150035	0.7630147825	2.21 E – 7
0.7	1.053018517	1.053009784	8.733 E – 6
0.8	1.366748495	1.366747768	7.27 E – 6
0.9	1.688423448	1.688420987	2.461 E – 6
1.0	2.0000000000	1.999989764	1.0236 E – 5

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Illustration III: Solve Completely the Differential Equation:

$$xy'' - (1+x)y' + y = 0$$

In the interval of [0 , 1] with $h = 0.1$, and given that $y_1(x) = e^x$ is one of the solutions to the differential equation. The analytical solution to the differential equation is

$$y_{(x)} = e^x - x - 1$$

Table III below shows the numerical computation to illustration III

TABLE III: COMPUTATIONAL RESULT TO ILLUSTRATION III

X	Y - EXACT	Y - COMPUTED	ERROR
0	0.0000000000	0.0000000000	0.0000000000
0.1	0.005170918076	0.005170908912	9.164E – 9
0.2	0.02140275800	0.02140196701	7.9099E – 7
0.3	0.04985880700	0.04985839210	4.149 E – 7
0.4	0.09182469700	0.09182402231	6.7469 E – 7
0.5	0.1487212700	0.1487193822	1.8878 E – 6
0.6	0.2221188000	0.2220654024	5.3397 E – 5
0.7	0.3137527070	0.3137519264	7.806 E – 7
0.8	0.4255409277	0.4255406792	2.485 E – 7
0.9	0.5596031114	0.5596027605	3.509 E – 7
1.0	0,7182818283	0.7182815932	2.351 E – 7

Illustration iv : Determine the general solution to the second order ordinary differential equation

$$2y'' + 7y' - 4y = 0$$

In the interval of [0, 1] with step length of $h = 0.1$ and given that $y_{(0)} = 2, and\ y'_{(0)} = 5$. This yields the analytical solution

$$y_{(x)} = 2.9e^{\frac{x}{2}} - 0.9e^{-4}$$

Table IV below shows the output of the computational result to the Illustration IV.

TABLE IV: COMPUTATIONAL RESULT TO ILLUSTRATION IV

X	Y -EXACT	Y – COMPUTED	ERROR
0.0	2.0000000000	1.999896132	1.03868 E – 4
0.1	2.445398138	2.445397066	1.072 E – 6
0.2	2.800599595	2.800598978	9.81 E – 7
0.3	3.098244513	3.098242976	1.537 E – 6
0.4	3.360361132	3.360361092	4.0 E – 8
0.5	3.601871956	3.601871891	6.20 E- 8
0.6	3.832944384	3.832944097	2.87 E – 7
0.7	4.060566835	4.060566584	2.51 E -7
0.8	4.289605640	4.289604463	1.177 E – 6
0.9	4.523513988	4.523513761	2.270 E – 7
1.0	4.764807610	4.764807594	1.60 E – 8

DISCUSSION OF RESULTS

From the foregoing, four illustrations were considered using the derived block method in which the computed results were compared with the analytical solution.

First and for most, Table I shows the computational error between the y – exact (analytical solution) in the interval [1, 2] with step length of $h = 0.1$ and the Y – computed (approximate solution using the newly derived method). It was observed that the error margin between the exact and approximate solution is very small. This shows that our newly derived method can as well be used to solve Second Order Ordinary Differential Equations without necessarily resolving to system of first order ordinary differential equations. For further affirmative on this statement, illustration II was equally examined.

Table II presented the error margin between the analytical solution and the approximate solution to illustration II. Critically examined the table of result, it was observed that there is little or no difference between the analytical solution and the approximate solution.

Taking a step further in testing for the accuracy of the block method derived, illustration III was presented, in which comparison was made between the exact solution and the approximate solution using the newly derived block method. The result shows that there is positive correlation between the analytical solution and the approximate solution using the newly derived block method. For further affirmation of the claims from illustrations I to III above, illustration IV was examined in which the solution to a Second Order Ordinary Differential Equation was sought for in the interval [0, 1]. The result presented in table IV shows the relationship between the analytical solution and the exact (theoretical solution). It could be observed from table IV that there is little or no difference between the exact solution and the approximate solution. This shows that the newly presented block method can as well be used in finding solution to Second Order Ordinary Differential Equations without necessarily resolving such an equation into a system of first Order Ordinary Differential Equations which is often laborious in many cases.

CONCLUSION AND RECOMMENDATION

At this juncture and having considered the output of the illustrations presented above, we are recommending unequivocally that this method, that is the Block off grid method for solving Second Order Ordinary Differential Equations can as well be used in solving Second Order Ordinary Differential Equations directly without necessarily resolving such an equation to a system of first Order Ordinary Differential Equations. Howbeit, there are still rooms for improvement in which methods of higher accuracy can be derived in the same interval that will give smaller error constant. Nonetheless, the method derived so far is equally recommended in finding direct solution to Second Order Ordinary Differential Equations.

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