International Journal of Mathematics and Statistics Studies
Vol.2, No.2, pp.43-53, June 2014
Published by European Centre for Research Training and Development UK (www.ea-journals.org)

GRONWALL – BELLMAN – OU-IANG TYPE INEQUALITIES

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ABSTRACT: *In this paper we prove some retarded nonlinear inequalities which provide explicit bounds on unknown function. Applications are also given to illustrate the usefulness of one of our results.*

Keywords: Integral Inequalities, Retarded Equation, Explicit Bounds, Boundedness.

INTRODUCTION

In the development of the theory of differential and integral equations integral inequalities which provide explicit bounds on unknown function take very important place. The literature on such inequalities is vast see [6, 8] and the references there in. One of the most useful inequalities in the development of theory of differential equation is given in following Lemma by Ou-Iang see [1].

Lemma. Let u and f are non-negative functions on $[0, \infty)$ satisfying

$$u^2(t) \leq k^2 + 2\int_0^t f(s)u(s)ds$$

for all , $t \in [0, \infty)$, where $k \geq 0$ is a constant then

$$u(t) \leq k + \int_0^t f(s)\,ds$$

for all $t \in [0, \infty)$.

In literature, the Ou-Iang type inequalities and their generalizations occurs which used effectively in the study of qualitative as well as quantitative properties of solutions of differential equations see [2-5, 7,9].In this paper we prove some new retarded inequalities which is generalization of above lemma and used to study the boundedness properties of solutions of integral and differential equations.

MAIN RESULTS

Theorem 2.1 : If u(x, y), m(x, y) are real valued nonnegative continuous functions defined for $x \geq 0$, $y \geq 0$, f(x, y, s, t) be continuous non-decreasing in x and y for each t, s. $0 \leq \alpha(x) \leq x$, $0 < \beta(y) \leq y$, $\alpha'(x)$, $\beta'(y) \geq 0$ are real valued continuous functions defined for $x \geq 0$, $y > 0$ satisfies

$$u^2(x, y) \leq m(x, y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x, y, s, t) u(s, t)\, dt\, ds \qquad \text{---- (2.1)}$$

then

$$u(x, y) \leq \sqrt{m(x, y)} + \frac{1}{2} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x, y, s, t) dt ds \qquad \text{---- (2.2)}$$

Proof . Let us assume that $m(x, y) > 0$. Fixed any numbers $\bar{x}$ and $\bar{y}$ with $0 \leq x \leq \bar{x}$ and $0 \leq y \leq \bar{y}$

Define a function

$$z(x, y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x, y, s, t) u(s, t) dt ds \qquad \text{---- (2.3)}$$

From (2.1) we get

$$u(x, y) \leq \sqrt{m(x, y) + z(x, y)} \qquad \text{---- (2.4)}$$

and $z(x, 0) = z(0, y) = 0$, clearly z (x, y) is positive non-decreasing function in each variable. Hence for $x \in [0, \bar{x}]$ and $y \in [0, \bar{y}]$ by direct calculation, we get

$$z_{xy}= f(x,y,\alpha(x),\beta(y))u(\alpha(x),\ \beta(y))\alpha'(x)\beta'(y)+\alpha'(x)\int_0^{\beta(y)}\partial_y\left(x,y,\alpha(x),t\right)u\left(\alpha(x),t\right)dt+$$

$$\beta'(y)\int_0^{\alpha(x)}\partial_x\left(x,y,s,\beta(y)\right)u\left(x,\beta(y)\right)ds+\int_0^{\alpha(x)}\int_0^{\beta(y)}\partial_x\,\partial_y\,f(x,y,s,t)\,u(s,t)\,dt\,ds$$

$$\leq f\left(x,y,\alpha(x),\beta(y)\right)\left(\sqrt{m(x,y)+z(x,y)}\right)\alpha'(x)\ \beta'(y)$$

$$+\alpha'(x)\int_0^{\beta(y)}\partial_y\,f\left(x,y,\alpha(x),t\right)\left(\sqrt{m\left(\alpha(x),t\right)+z\left(\alpha(x),t\right)}\right)$$

$$+\beta'(y)\int_0^{\alpha(x)}\partial_x\,f\left(x,y,s,\beta(y)\right)\left(\sqrt{m\left(s,\beta(y)\right)+z\left(s,\beta(y)\right)}\right)ds$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)}\partial_x\partial_y\,f\left(x,y,s,t\right)\left(\sqrt{m\left(s,t\right)+z\left(s,t\right)}\right)dt\,ds$$

$$z_{xy}\leq\sqrt{m\left(\overline{x},\overline{y}\right)+z\left(x,y\right)}\Big\{f(x,y,\alpha(x),\ \beta(y))\alpha'(x)\beta'(y)+\alpha'(x)\int_0^{\beta(y)}\partial_y f\left(x,y,\alpha(x),t\right)dt+\beta'(y)\int_0^{\alpha(x)}\partial_x\,f\left(x,y,s,\beta,(y\right)ds$$
$$+\int_0^{\alpha(x)}\int_0^{\beta(y)}\partial_x\,\partial_y\,f\left(x,y,s,t\right)dt\,ds\Big\}$$

$$\frac{z_{xy}\left(x,y\right)}{\sqrt{m\left(\overline{x},\overline{y}\right)+z\left(x,y\right)}}\leq\frac{\partial^2}{\partial_x\,\partial_y}\left[\int_0^{\alpha(x)}\int_0^{\beta(y)}f\left(x,y,s,t\right)dt\,ds\right]$$

on the other hand

$$\frac{\partial}{\partial y}\left(\frac{z_x\left(x,y\right)}{\sqrt{m\left(\overline{x},\overline{y}\right)+z\left(x,y\right)}}\right)\leq\frac{\partial^2}{\partial_x\,\partial_y}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}f\left(x,y,s,t\right)dt\,ds\right)\qquad\text{---- (2.5)}$$

Keeping x fixed in (2.5), integrating with respect to y from 0 to y, and using fact that z_x (x, 0) = 0, we deduce.

$$\frac{z_x(x,y)}{\sqrt{m(\bar{x},\bar{y})+z(x,y)}} \le \frac{\partial}{\partial x}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds\right) \quad \text{---- (2.6)}$$

Now keeping y fixed in (2.6) and integrating with respect to x from 0 to x we get

$$\sqrt{m(\bar{x},\bar{y})+z(x,y)} \le \sqrt{m(\bar{x},\bar{y})} + \frac{1}{2}\int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds$$

for (x, y) $\in [0, \bar{x}] \times [0, \bar{y}]$

let x = $\bar{x}$ and y = $\bar{y}$ in above inequality we get

$$\sqrt{m(\bar{x},\bar{y})+z(\bar{x},\bar{y})} \le \sqrt{m(\bar{x},\bar{y})} + \frac{1}{2}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(\bar{x},\bar{y},s,t)\,dt\,ds$$

As $\bar{x}$ and $\bar{y}$ are arbitrarily and u(x, y) ≤ $\sqrt{m(x,y)+z(x,y)}$ we have obtain inequality in (2.2).

Corollary 2.1. . Let m, n, g $\in C(R_+ \times R_+, R_+)$, α , $\beta \in C^1(R_+, R_+)$ and let α, β is non-decreasing with α(x) ≤ x, β(y) ≤ y if u $\in C(R_+ \times R_+, R_+)$ Satisfies

$$u(x, y) \le \sqrt{m} + \frac{1}{2}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds, \quad x \ge 0,\ y \ge 0$$

Theorem 2.2 . Let m, n, g $\in C(R_+ \times R_+, R_+)$, α , $\beta \in C^1(R_+, R_+)$ and let α, β is non-decreasing with α(x) ≤ x, β(y) ≤ y if u $\in C(R_+ \times R_+, R_+)$ Satisfies

$$u^2(x,y) \le m(x,y) + n(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\,u(s,t)\,dt\,ds \qquad \text{---- (2.7)}$$

then when $x \ge 0$, $y \ge 0$

$$u(x,y) \le \sqrt{m(x,y)} + \frac{1}{2}n(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\,dt\,ds \qquad \text{---- (2.8)}$$

Proof . Let us assume that $m(x, y) > 0$. Fixed any number $\bar{x}$ and $\bar{y}$ with $0 \le x \le \bar{x}$ and $0 \le y \le \bar{y}$

$$z(x,y) = m(\bar{x},\bar{y}) + n(\bar{x},\bar{y})\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\,u(s,t)\,dt\,ds$$

Then $z(x, y) > 0$, $z(0, y) = z(x, 0) = m(\bar{x},\bar{y})$ and from (2.7) we get

$$u^2(x,y) \le z(x,y) \text{ i.e. } u(x,y) \le \sqrt{z(x,y)} \qquad \text{---- (2.9)}$$

Clearly $z(x, y)$ is positive non-decreasing function in each variable. Hence we get

$$\mathbf{z_{xy}} = \mathbf{n}(\bar{x},\bar{y})\, g(\alpha(x),\beta(y))\,.\,u(\alpha(x),\beta(y))\,.\,\alpha'(x).\beta'(y)$$

$$\le n(\bar{x},\bar{y})\, g(\alpha(x),\beta(y))\sqrt{z(\alpha(x),\beta(y))}\,\alpha'(x).\beta'(y)$$

$$\frac{z_{xy}}{\sqrt{z(x,y)}} \le n(\bar{x},\bar{y})\, g(\alpha(x),\beta(y))\,\alpha'(x)\beta'(y)$$

i.e.

$$\frac{\partial}{\partial_y}\left(\frac{z_x}{\sqrt{z(x,y)}}\right) \le n(\bar{x},\bar{y})\, g\big(\alpha(x),\ \beta(y)\big)\,\alpha'(x)\,\beta'(y)$$

Keeping x fixed, integrating above inequality with respect to y from 0 to y and using fact that z_x (x, 0) = 0, we get

$$\frac{z_x(x,y)}{\sqrt{z(x,y)}} \le n(\bar{x},\bar{y}) \int_0^y g\big(\alpha(x),\ \beta(t)\big)\,\alpha'(x)\,\beta'(t)\,dt$$

Now keeping y fixed in above inequality and integrating with respect to x from 0 to x, we deduce

$$\sqrt{z(x,y)} \le \sqrt{m(\bar{x},\bar{y})} + \frac{n(\bar{x},\bar{y})}{2} \int_0^x \int_0^y g\big(\alpha(s),\ \beta(t)\big)\,\alpha'(s)\,\beta'(t)\,ds\,dt$$

By making change of variable on right-hand side of above inequality, we get

$$\sqrt{z(x,y)} \le \sqrt{m(\bar{x},\bar{y})} + \frac{1}{2} n(\bar{x},\bar{y}) \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t)\,dt\,ds$$

Taking x = $\bar{x}$, y = $\bar{y}$ in the inequality (2.10) , since $\bar{x}$ and $\bar{y}$ are arbitrary and as u(x, y) ≤ $\sqrt{z(x,y)}$ we get the required inequality in (2.8).

Theorem 2.3 . Assume m, f, α, β be as in Theorem 2.1 ω(u) is positive continuous non-decreasing for u > 0 with ω (0) = 0 and $\int_1^\infty \frac{ds}{\omega(\sqrt{s})} = \infty$ if u ∈ C($R_+ \times R_+$, R_+) satisfies

$$u^2(x,y) \le m(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x,y,s,t)\,\omega\big(u(s,t)\big)\,dt\,ds,\ x \ge 0,\ y \ge 0 \qquad \text{---- (2.11)}$$

then

$$u(x, y) \le G^{-1}\left(G\left(\sqrt{m(x,y)}\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds \right) \qquad \text{---- (2.12)}$$

where $G(r) = \int_1^r \frac{ds}{\omega\left(\sqrt{s}\right)}$, $r \ge 0$

Proof : Assume $\bar{x}, \bar{y} > 0$ be fixed and let

$$z(x, y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} f(x,y,s,t)\,\omega\left(u(s,t)\right) dt\,ds$$

Clearly z (x, y) is non-decreasing for x and y hence for $x \in [0, \bar{x}]$, $y \in [0, \bar{y}]$, we have

$$z_{xy} = f\left(x,y,\alpha(x),\beta(y)\right) \omega\left(u(\alpha(x), \beta(y)\right) \alpha'(x)\,\beta'(y) + \beta'(y) \int_0^{\alpha(x)} \partial_x f\left(x,y,s,\beta(y)\right) \omega\left(u(s,\beta(y))\right) ds$$

$$+ \alpha'(x) \int_0^{\beta(y)} \partial_y f\left(x,y,\alpha(x),t\right) \omega\left(u(\alpha(x),t)\right) dt + \int_0^{\alpha(x)} \int_0^{\beta(y)} \partial_x \partial_y f\left(x,y,s,t\right) \omega\left(u(s,t)\right) dt\,ds$$

$$\le f\left(x,y,\alpha(x),\beta(y)\right) \omega\left(\sqrt{m\left(\alpha(x),y\right) + z(x,y)}\right).\ \alpha'(x)\,\beta'(y)$$

$$+ \alpha'(x) \int_0^{\beta(y)} \partial_y f\left(x,y,\alpha(x),t\right) \omega\left(\sqrt{m\left(\alpha(x),t\right) + z\left(\alpha(x),t\right)}\right) dt$$

$$+\ \beta'(y)\int_0^{\alpha(x)}\partial_x f\left(x,y,s,\beta(y)\right)\omega\left(\sqrt{m\left(s,\beta(y)\right)+z\left(\beta(y)\right)}\right)ds$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)}\partial_x\,\partial_y\,f\left(x,y,s,t\right)\omega\sqrt{m(s,t)+z(s,t})dt\,ds$$

$$\le\omega\left(\sqrt{m\left(\bar{x},\bar{y}\right)+z\left(x,y\right)}\right)$$

$$\Big[f\left(x,y,\alpha(x)\right)\beta(y).\alpha'(x).\beta'(y)+\alpha'(x)\int_0^{\beta(y)}\partial_y f\left(x,y,\alpha,(x),t\right)dt+\beta'(y)$$

$$\int_0^{\alpha(x)}\partial_x f\left(x,y,z,\beta(y)\right)ds+\int_0^{\alpha(x)}\int_0^{\beta(y)}\partial_x\partial_y f(x,y,s,t)dtds\Big]$$

i.e .
$$\frac{z_{xy}}{\omega\left(\sqrt{m\left(\bar{x},\bar{y}\right)+z\left(x,y\right)}\right)}\le\frac{\partial^2}{\partial_x\partial_y}\int_0^{\alpha(x)}\int_0^{\beta(y)}f(x,y,s,t)\,dt\,ds$$

We have

$$\frac{\partial}{\partial_y}\left(\frac{z_x}{\omega\left(\sqrt{m\left(\bar{x},\bar{y}\right)+z\left(x,y\right)}\right)}\right)\le\frac{\partial^2}{\partial_x\,\partial_y}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}f(x,y,s,t)\,dt\,ds\right)$$

Integrating both sides of above inequailty with respect to y from 0 to y we deduce

$$\frac{z_x}{\omega\left(\sqrt{m\left(\bar{x},\bar{y}\right)+z\left(x,y\right)}\right)}\le\frac{\partial}{\partial_x}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}f(x,y,s,t)\,dt\,ds\right)$$

Integrating above inequality with respect to x from 0 to x and using definition of G, we get

$$G\left(\sqrt{m(\bar{x},\bar{y})+z(x,y)}\right) \le G\left(\sqrt{m(\bar{x},\bar{y})}\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds \quad \text{----}$$

(2.13)

for $x \in [0,\bar{x}], y \in [0,\bar{y}]$.

As $\int_1^{\infty} \frac{ds}{\omega(\sqrt{s})} = \infty$ from (2.13) we have

$$\sqrt{m(\bar{x},\bar{y})+z(x,y)} \le G^{-1}\left(G\left(\sqrt{m(\bar{x},\bar{y})}\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(x,y,s,t)\,dt\,ds \right)$$

let $x = \bar{x}$, $y = \bar{y}$ in above inequality, we get

$$\sqrt{m(\bar{x},\bar{y})+z(\bar{x},\bar{y})} \le G^{-1}\left(G\left(\sqrt{m(\bar{x},\bar{y})}\right) + \int_0^{\alpha(\bar{x})}\int_0^{\beta(\bar{y})} f(\bar{x},\bar{y},s,t)\,dt\,ds \right)$$

As $\bar{x},\bar{y}$ are arbitrarily and u(x, y) ≤ $\sqrt{m(x,y)+z(x,y)}$ we obtain inequality in (2.12).

Theorem 2.4 : Suppose m, n, g $\in C(R_+ \times R_+, R_+)$, $\alpha,\beta \in C^1(R_+, R_+)$ and m, n, g, α, β are non-decreasing function with α(x) ≤ x, β(y) ≤ y for x ≥ 0, y ≥ 0 assume $\omega \in C(R_+, R)$ be a non-decreasing function such that $\omega(t) > 0$ for t > 0 and $\int_1^{\infty} \frac{ds}{\omega(\sqrt{s})} = \infty$ if

$u \in C(R_+ \times R_+, R_+)$ satisfies

$$u(x,y) \le m(x,y) + n(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(t,s)\omega(u(s,t))\,dt\,ds \quad x \ge 0,\ y \ge 0 \quad \text{----}$$

(2.14)

$$u(x, y) \leq G^{-1}\left(G\left(\sqrt{m(x,y)}\right) + \frac{1}{2}n(x,y)\right)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\,dt\,ds\ , \quad x \geq 0,\ y \geq 0 \qquad \text{----}$$

(2.15)

where

$$G(r) = \int_1^r \frac{ds}{\omega\left(\sqrt{s}\right)}, r \geq 0$$

Proof : Let us assume that m (x, y) > 0, Fixed any number $\bar{x}$ and $\bar{y}$ with $\leq x < \bar{x}$, $0 \leq y \leq \bar{y}$.

Define a function z(x, y) as

$$z(x, y) = m(\bar{x},\bar{y}) + n(\bar{x},\bar{y}) \int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\,\omega\left(u(s,t)\right)dt\,ds$$

Then z (x, y) > 0, z (0, y) = z (x, 0) = m $(\bar{x},\bar{y})$ and from (2.14) we have

$u^2(x, y) \leq z(x, y)$ implies $u(x, y) \leq \sqrt{z(x,y)}$ clearly z (x, y) is positive non-decreasing function in each variable.

Hence we get

$$z_{xy} = n(\bar{x},\bar{y})\,g\left(\alpha(x),\beta(y)\right)\omega(u\left(\alpha(x),\beta(y)\right)\ \alpha'(x)\ \beta'(y)$$

$$\leq n(\bar{x},\bar{y})\,g\left(\alpha(x),\beta(y)\right)\omega\left(\sqrt{z\left(\alpha(x),\beta(y)\right)}\right)\alpha'(x)\ \beta'(y)$$

$$\frac{z_{xy}}{\omega\left(\sqrt{z(x,y)}\right)} < n(\bar{x},\bar{y})\,g\left(\alpha(x),\beta(y)\right)\alpha'(x)\beta'(y)$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{z_x}{\omega\sqrt{(z,y)}}\right)\leq n\left(\bar{x},\bar{y}\right)g\left(\alpha(x),\beta(y)\right)\alpha'(x)\beta'(y)$$

Keeping x fixed, integrating above inequality with respect to y from 0 to y and using fact that Zx (x, 0) = 0 we get

$$\frac{z_x\left(x,y\right)}{\omega\sqrt{z\left(x,y\right)}}\leq n\left(\bar{x},\bar{y}\right)\int_0^y g\left(\alpha(x),\beta(y)\right)\alpha'(x)\beta'(t)\,dt$$

Keeping y fixed in above inequality and integrating with respect to x from 0 to x we get

$$G\left(\sqrt{z(x,y)}\right)\leq G\left(\sqrt{m(\bar{x},\bar{y})}\right)+n\left(\bar{x},\bar{y}\right)\int_0^x\int_0^y g\left(\alpha(s),\beta(t)\right)\alpha'(s)\beta'(t)\,dt\,ds$$

By making change of variable on right-hand side of above inequality, we get

$$\sqrt{z(x,y)}\leq G^{-1}\left(G\left(\sqrt{m(\bar{x},\bar{y})}\right)+n\left(\bar{x},\bar{y}\right)\int_0^{\alpha(x)}\int_0^{\beta(y)} g\left(\alpha(s,t),dtds\right)\right)$$

for $x\in[0,\bar{x}],y\in[0,\bar{y}]$

let x = $\bar{x}$, y = $\bar{y}$ in (2.16) we have

$$\sqrt{z\left(\bar{x},\bar{y}\right)}<G^{-1}\left(G\left(\sqrt{m\left(\bar{x},\bar{y}\right)}\right)+n\left(\bar{x},\bar{y}\right)\int_0^{\alpha(x)}\int_0^{\beta(y)} g\left(s,t\right)dt\,ds\right)$$

As $\bar{x},\bar{y}$ are arbitrarily and u (x, y) ≤ $\sqrt{z(x,y)}$ we get required inequality in (2.15).

APPLICATION

In this section we will show that our one of the result is used in proving the boundedness of solutions of nonlinear integral equation. Consider the nonlinear partial integral equation of the form

$$u^2(x,y) = h(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} G(x,y,s,t,u(s,t))\,dt\,ds \qquad \text{---- (3.1)}$$

where all the functions are continuous on their respective domains of their definitions and

$$|h(x,y)| \le m(x,y) \qquad \text{---- (3.2)}$$

$$|F(x,y,t,s,u)| \le f(x,y,t,s)|u| \qquad \text{----- (3.3)}$$

for $x \ge 0$, $y \ge 0$ where m, f, α, β be as in Theorem (2.1). Using (3.2), (3.3) in (3.1) and using Theorem 2.1 we obtain the bound on the solution u(x, y) of (3.1).

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