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GENERALIZED SECOND GRADE FLUID PERFORMING SINUSOIDAL MOTION IN AN INFINITE CYLINDER

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ABSTRACT: *This paper shows the calculation of velocity field and shear stress corresponding to Generalized second grade fluid performing sinusoidal motion. Shear stress is found by using* $D_t^{\beta}\ G_{a,b,c}(.,.) = G_{a,b+\beta,c}(.,.)$ *[4]. Velocity field obtained by applying Laplace and Hankel transforms. The solution have been written in series form by using generalised function G.,.,.(.,t) and Bessel functions.*

KEYWORDS: Velocity Field, Generalized Second Grade Fluid, Shear Stress

INTRODUCTION

The exact solution corresponding to the flow of fractional second grade fluid in circular cylinder were found and written under integral and series form by using G.,.,.(.,t) function and found Newtonian and ordinary second grade fluid performing the same motion [1]. The aim of this paper is to calculate shear stress corresponding to non-Newtonian fluid by applying fractional derivative and the result mentioned in abstract of this paper.

The non-Newtonian fluids with fractional derivatives have encountered a lot of success in describing complex fluid dynamics. The governing equations corresponding to the motion of fluid are obtained from those of ordinary fluids by replacing inner time derivative by the so called Riemann Liouville operator D^{β}_{t} defined by

$$D_t^{\beta}\, f(t) = \frac{1}{\Gamma_{1-\beta}} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta}}\, d\tau \qquad 0 \le \beta < 1$$

$$= \frac{d}{dt} f(t) \qquad \beta = 1$$

An excellent discussion of fractional differential equation and a good history of fractional calculus is given by K. S. Miller [9] carl. F. Lorenzo [3] presented a very useful paper on fractional derivative which are much flexible in describing visco elastic behaviour of fluids [6] M. Kamran [7] and A. Mahmood [8] presented exact solution for unsteady rotational flow of the generalized second grade fluid. M. Athar [2] solved Taylor–Couette flow of the generalized second grade fluid.

Governing Equations

In this paper we consider the velocity **V** and the extra stress **S** of the form

$$V = V(r,t) = \omega(r,t) e_{\theta}$$

$$S = S(r,t)$$

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Where e_θ is the unit vector in the θ direction of the cylindrical coordinate system. At $t = 0$ we have

$$\omega(r,0) = 0$$

The governing equations corresponding to such motion of ordinary second grade fluid are

$$\tau(r,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})(\frac{\partial}{\partial r} - \frac{1}{r})\, \omega(r,t) \tag{1}$$

$$\frac{\partial \omega(r,t)}{\partial t} = (\vartheta + \frac{\alpha \partial}{\partial t})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\, \omega(r,t) \tag{2}$$

Where μ is dynamic viscosity of the fluid and $\alpha = \frac{\alpha_1}{\rho}$ is material constant; $\vartheta = \frac{\mu}{\rho}$ Kinematic viscosity of the fluid. Where ρ being its constant density and $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress.

Governing equations corresponding to fractional second grade fluids are obtained by replacing inner time derivative w.r.t "t" by fractional derivative D_t^β , $\beta > 0$.

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)(\frac{\partial}{\partial r} - \frac{1}{r})\, \omega(r,t) \tag{3}$$

$$\frac{\partial \omega(r,t)}{\partial t} = (\vartheta + \alpha_1 D_t^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\, \omega(r,t) \tag{4}$$

Flow through a circular cylinder with a shear on boundary

Consider incompressible generalized second grade fluid at rest, in an infinitely long cylinder of radius R>0. At time t=0 fluid is at rest and at time $t=0^+$ cylinder begins to rotate and boundary of cylinder applies a sinusoidal shear stress on fluid. The fluid is gradually moved. The governing equations as given by (3) and (4). Boundary conditions and initial conditions are

$$\omega(r,0) = 0 \qquad \text{Where } r\epsilon(0,R]$$

$$\tau(R,t) = (\mu + \alpha_1 D_t^\beta)(\frac{\partial}{\partial r} - \frac{1}{r})\, \omega(r,t)\,\Big|_{(R=r)}$$

$$= \Omega R sin(\omega,t) \text{ with } t>o$$

$$\Omega \text{ is constant}$$

Calculations of velocity field

Applying Laplace transform to (3) and (4), and then applying Hankel transform and breaking $\bar{\omega}_H\,(r_n,q)$ into two parts [1].

$$\bar{\omega}_{1H}\,(r_n,q) = \frac{1}{\mu r_n^2}\left(\frac{RJ_1(Rr_n)\Omega\omega R}{(q^2+\omega^2)}\right) \tag{5}$$

and

$$\bar{\omega}_{2H}\,(r_n,q) = -\frac{RJ_1(Rr_n)\Omega\omega Rq(1+q^{\beta-1}\alpha r_n^2)}{\mu r_n^2(q^2+\omega^2)(q+\upsilon r_n^2+\alpha q^\beta r_n^2)} \tag{6}$$

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Applying inverse Hankel transform to (5) and (6) and then inverse Laplace transform we get the velocity field;

$$\omega(r,t) = \frac{\Omega r^3}{2\mu R}\sin(\omega t) \; - \; 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)\Omega}{\mu r_n^2 J_1(Rr_n)} \times$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{1-\beta,1-\beta-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds +$$

$$\alpha r_n^2\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{1-\beta,-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds] \tag{7}$$

Where generalized function $G_{a,b,c}$ (d,t) is defined by [3] equations (97) and (101)

$$G_{a,b,c}(\mathrm{d},\mathrm{t}) = L^{-1}\,[\frac{q^b}{(q^a-d)^c}] =$$

$$\sum_{k=0}^{\infty}\frac{d^k\Gamma_{c+k}}{\Gamma_c\Gamma_{k+1}} \cdot \frac{t^{(c+k)a-b-1}}{\Gamma(c+k)a-b}$$

$$Re(ac-b) > 0\,, \quad \left|\frac{d}{q^a}\right| < 1$$

Calculation of shear stress

$$\tau(r,t) = (\mu + \alpha_1 D_t^{\beta})(\frac{\partial}{\partial r} \; - \; \frac{1}{r})\,\omega(r,t)$$

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^{\beta}\right)\frac{\Omega r^2}{\mu R}\sin(\omega t) + 2\sum_{n=1}^{\infty}\frac{J_2(rr_n)\Omega}{\mu r_n J_1(Rr_n)} \times$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{1-\beta,1-\beta-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds +$$

$$\alpha r_n^2\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{1-\beta,-\beta k,k+1}(-\alpha r_n^2, t-s)ds]$$

$$= \frac{\Omega r^2\ \sin(\omega t)}{R} + 2\sum_{n=1}^{\infty}\frac{J_2(rr_n)\Omega}{(r_n)J_1(Rr_n)} \times$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{1-\beta,1-\beta-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds +$$

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$$\alpha\mu r_n^2 \sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{1-\beta,-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds] +$$

$$\frac{\alpha_1 \omega r^2}{\mu R}[D_t^{\beta}(\sin(\omega t)] + \quad 2\,\alpha_1 \sum_{n=1}^{\infty} \frac{J_2(r r_n)\Omega}{\mu r_n J_1(R r_n)} \times$$

$$(\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{1-\beta,1-\beta k,k\ +1}(-\alpha r_n^2, t-s)ds +$$

$$\alpha_1 \alpha r_n^2 \sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{1-\beta,\beta-\beta k,k+1}(-\alpha r_n^2, t-s)ds]) \qquad (8)$$

1. Newtonian

Applying $\alpha \to 0$ and $\alpha_1 \to 0$ in (7 and 8) we get,

$$\omega(r,t) = \frac{\Omega r^3}{2\mu R}\sin(\omega t) \ - \ 2\sum_{n=1}^{\infty} \frac{J_1(r r_n)\Omega}{\mu r_n^2 J_1(R r_n)} \mathrm{x}$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{1-\beta,1-\beta-\beta k,k+1}(0, t-s)\, ds \qquad (9)$$

$$\tau(r,t) = \frac{\Omega r^2 \sin(\omega t)}{R} + \quad 2\sum_{n=1}^{\infty} \frac{J_2(r r_n)\Omega}{r_n J_1(R r_n)} \times$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{1-\beta,1-\beta-\beta k,k\ +1}(0, t-s)ds] \qquad (10)$$

$\beta \to 1$ for ordinary second grade fluid

$$\omega(r,t) = \frac{\Omega r^3}{2\mu R}\sin(\omega t) \ - \ 2\sum_{n=1}^{\infty} \frac{J_1(r r_n)\Omega}{\mu (r_n^2) J_1(R r_n)} \mathrm{x}$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{0,-k,k\ +1}(0, t-s)ds] \qquad (11)$$

$$\tau(r,t) = \frac{\Omega r^2 \sin(\omega t)}{R} \ \mathrm{x} \quad 2\sum_{n=1}^{\infty} \frac{J_2(r r_n)\Omega}{r_n J_1(R r_n)} \times$$

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$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{0,-k,k\ +1}(0, t-s) ds] \tag{12}$$

CONCLUSION

The velocity field and shear stress corresponding to generalized second grade fluid were calculated and written in the series form with the help of generalized function G.,.,.(.,t). The velocity field was calculated by applying Laplace transform and Hankel transform. Shear stress was calculated by using

$$D_t^{\beta} G_{a,\, b,\, c}\, (.,\, t) = G_{a,\, b+\beta,\, c}\, (.,\, t)$$

Newtonian and ordinary second grade fluid were found as a limiting case.

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