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EXTENDED POWER LINDLEY DISTRIBUTION: A NEW STATISTICAL MODEL FOR NON-MONOTONE SURVIVAL DATA

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ABSTRACT: *A new statistical model for non-monotone survival data is proposed with some of its statistical properties as an extension of power Lindley distribution. These include the density and hazard rate functions with their behavior, moments, moment generating function, skewness, kurtosis measures, and quantile function. Maximum likelihood estimation of the parameters and their estimated asymptotic distribution and confidence intervals are derived. Rényi entropy as a measure of the uncertainty in the model is derived. An application of the model to a real data set is presented and compared with the fit attained by other well-known existing distributions.*

KEYWORDS: Extended power Lindley distribution; power Lindley distribution; non-monotone survival data

INTRODUCTION

The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields, such as public health, actuarial science, biomedical studies, demography, and industrial reliability. The failure behavior of any system can be considered as a random variable due to the variations from one system to another resulting from the nature of the system. Therefore, it seems logical to find a statistical model for the failure of the system. In other applications, survival data are categorized by their hazard rate, e.g., the number of deaths per unit in a period of time. Survival data are categorized by their hazard rate which can be monotone (non-increasing and non-decreasing) or non-monotone (bathtub and upside-down bathtub, or unimodal). For modeling of such survival data, many models have been proposed based on hazard rate type. Among these, Weibull distribution has been used extensively in survival studies, but it does not fit data with a non-monotone hazard rate shape. A one-parameter distribution was introduced by Lindley (1958) as an alternative model for data with a non-monotone hazard rate shape. This model becomes the well-known Lindley distribution. Properties and applications of Lindley distribution in reliability analysis were studied by Ghitany et al. (2008) showing that this distribution may provide a better fit than the exponential distribution. Some researchers have proposed new classes of distributions based on modifications of the Lindley distribution, including their properties and applications. Recently, several authors—including Zakerzadeh and Dolati (2009), Nadarajah et al. (2011), Elbatal et al. (2013), Ashour and Eltehiwy (2014), and Oluyede and Yang (2015)—proposed and generalized Lindley distribution with its mathematical properties and applications. A discrete version of Lindley distribution has been suggested by Deniz and Ojeda (2011) with applications in count data related to insurance. A new extension of Lindley distribution, called extended Lindley (EL) distribution, which offers a more flexible model for lifetime data, was introduced by Bakouch et al. (2012). Shanker et al. (2013) proposed a two-parameter Lindley distribution for modeling waiting and survival time data.

The power Lindley (PL) distribution with its inference was proposed by Ghitany et al. (2013) and generalized by Liyanage and Pararai (2014). Estimation of the reliability of a stress-strength system from power Lindley distribution was discussed by Ghitany et al. (2015). The inverse Lindley distribution with application to head and neck cancer data was introduced by Sharma et al. (2015). In this paper we introduce a new type of Lindley distribution with three parameters as an extension to power Lindley distribution.

The Lindley distribution has been proposed by Lindley (1958) in the context of Bayes' theorem as a counter example of fiducial statistics with the following probability density function (pdf)

$$f(y;\theta)=\frac{\theta^2}{\theta+1}(1+y)e^{-\theta y};\theta,y>0. \qquad (1)$$

Ghitany et al. (2008) discussed Lindley distribution and its applications extensively and showed that the Lindley distribution fits better than the exponential distribution based on the waiting time at a bank for service.

Shanker et al. (2013) proposed two-parameter extensions of Lindley distribution with the following pdf

$$f(z;\theta)=\frac{\theta^2}{\theta+\beta}(1+\beta z)e^{-\theta z};\theta,\beta,z>0. \qquad (2)$$

This gives better fitting than the original Lindley distribution. Another two-parameter Lindley distribution was introduced by Ghitany et al. (2013) named "power Lindley distribution" using the transformation $X=Y^{\frac{1}{\alpha}}$, with pdf

$$f(x;\theta)=\frac{\alpha\theta^2}{\theta+1}(1+x^{\alpha})\,x^{\alpha-1}e^{-\theta x^{\alpha}};\theta,\alpha,x>0.$$

where Y is a random variable having pdf (1).

Using the transformation $X=Z^{\frac{1}{\alpha}}$, where Z has the pdf (2), we introduce a more flexible distribution with three parameters called "extended power Lindley distribution (EPL)," which gives us a better performance in fitting non-monotonic survival data.

The aim of this paper is to introduce a new Lindley distribution with its mathematical properties. These include the shapes of the density and hazard rate functions, the moments, moment generating function and some associated measures, the quantile function, and stochastic orderings. Maximum likelihood estimation of the model parameters and their asymptotic standard distribution and confidence interval are derived. Rényi entropy as a measure of the uncertainty in the model is derived. Finally, application of the model to a real data set is presented and compared to the fit attained by some other well-known Lindley distributions.

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The extended power Lindley distribution

An extended power Lindley distribution with parameters θ, β, and α is defined by its probability density function and cumulative distribution function according to the following definition.

Definition: Let Z be a random variable having pdf (2), then the random variable $X = Z^{\frac{1}{\alpha}}$ is said to follow an EPL distribution pdf

$$\mathrm{f}(x;\theta,\beta,\alpha) = \frac{\alpha\theta^2}{\theta+\beta}(1+\beta x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}};\theta,\beta,\alpha,x>0 \tag{3}$$

and cumulative distribution function (cdf)

$$\mathrm{F}(x;\theta,\beta,\alpha) = 1-\left(1+\frac{\theta\beta}{\theta+\beta}x^{\alpha}\right)e^{-\theta x^{\alpha}};\theta,\beta,\alpha,x>0. \tag{4}$$

Remark: The pdf (3) can be shown as a mixture of two distributions as follows:

$$f(x;\theta,\beta,\alpha) = \mathrm{pf}_1(x)+(1-p)\mathrm{f}_2(x)$$

where

$$p = \frac{\theta}{\theta+\beta}, f_1(x) = \alpha\theta x^{\alpha-1}e^{-\theta x^{\alpha}}, \mathrm{x}>0 \text{ and } f_2(x) = \alpha\theta^2 x^{2\alpha-1}e^{-\theta x^{\alpha}}, \mathrm{x}>0.$$

We see that the EPL is a two-component mixture of Weibull distribution (with shape α and scale θ), and a generalized gamma distribution (with shape parameters $2,\alpha$ and scale θ), with mixing proportion $p = \theta/(\theta+\beta)$.

We use $X \sim EPL(\theta,\beta,\alpha)$ to denote the random variable having extended power Lindley distribution with parameters θ,β,α with cdf and pdf in (3) and (4) , respectively.

The behavior of $f(x)$ at $x=0$ and $x=\infty$, respectively, are given by

$$f(0) = \begin{cases} \infty, & \text{if } 0<\alpha<1 \\ \dfrac{\theta^2}{\theta+\beta}, & \text{if } \alpha=1, \\ 0, & \text{if } \alpha>1, \end{cases} \quad f(\infty)=0.$$

The derivative of $f(x)$ is obtained from (3) as

$$f'(x) = \frac{\alpha\theta^2}{\theta+\beta}x^{\alpha-2}e^{-\theta x^{\alpha}}\psi(x^{\alpha}),\ x>0,$$

where

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$$\psi(y) = ay^2 + by + c,\ y = x^{\alpha},$$

with

$$a = -\theta\beta\alpha,\ b = \beta(2\alpha - 1) - \theta\alpha,\ c = \alpha - 1.$$

Clearly, $f'(x)$ and $\psi(y)$ have the same sign. $\psi(y)$ is decreasing quadratic function (unimodal with maximum value at the point $y = -\frac{b}{2a}$) with $\psi(0) = c$ and $\psi(\infty) = -\infty$. If $\alpha = \beta = 1$, i.e., the case of Lindley distribution, f(x) is decreasing (unimodal) if $\theta \geq 1$ with $f(0) = \frac{\theta^2}{\theta + 1}$ and f(∞)=0.

Fig. 1 shows the pdf of the EPL distribution for some values of θ, β, and α displaying the behavior of $f(x)$.

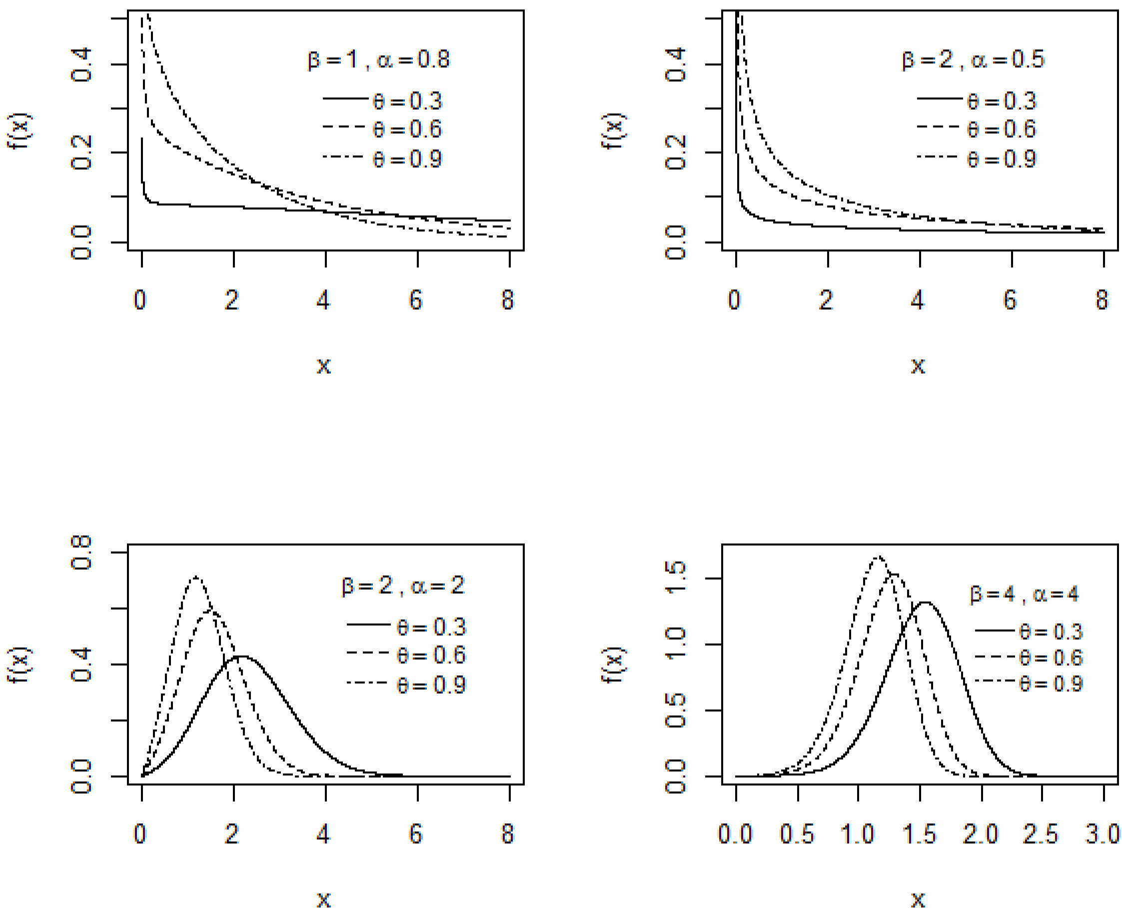

Fig. 1. Plots of the probability density function of the EPL distribution for different values of θ, β, and α.

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Survival and hazard functions

The survival and hazard rate functions of the EPL distribution are given respectively by

$$s(x) = 1 - F_X(x) = \left(1 + \frac{\theta\beta}{\theta+\beta} x^{\alpha}\right) e^{-\theta x^{\alpha}}; \theta, \beta, \alpha, x > 0, \qquad (5)$$

and

$$h(x) = \frac{f(x)}{s(x)} = \frac{\alpha\theta^2 x^{\alpha-1}(1+\beta x^{\alpha})}{\theta+\beta+\beta\theta x^{\alpha}}; \theta, \beta, \alpha, x > 0. \qquad (6)$$

The behavior of $h(x)$ in (6) of the $EPL(\theta, \beta, \alpha)$ distribution at $x = 0$ and $x = \infty$, respectively, are given by

$$h(0) = \begin{cases} \infty, & \text{if } 0 < \alpha < 1 \\ \dfrac{\theta^2}{\theta+\beta}, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1, \end{cases} \qquad h(\infty) = \begin{cases} 0, & \text{if } \alpha < 1, \\ \theta, & \text{if } \alpha = 1, \\ \infty, & \text{if } \alpha > 1. \end{cases}$$

It can be seen that when $\alpha = \beta = 1$, i.e., in the case of Lindley distribution, $h(x)$ is increasing for all $\theta > 0$ with $h(0) = \dfrac{\theta^2}{\theta+1}$ and $h(\infty) = \theta$.

Fig. 2 shows $h(x)$ of the EPL distribution for some choices of θ, β, and α.

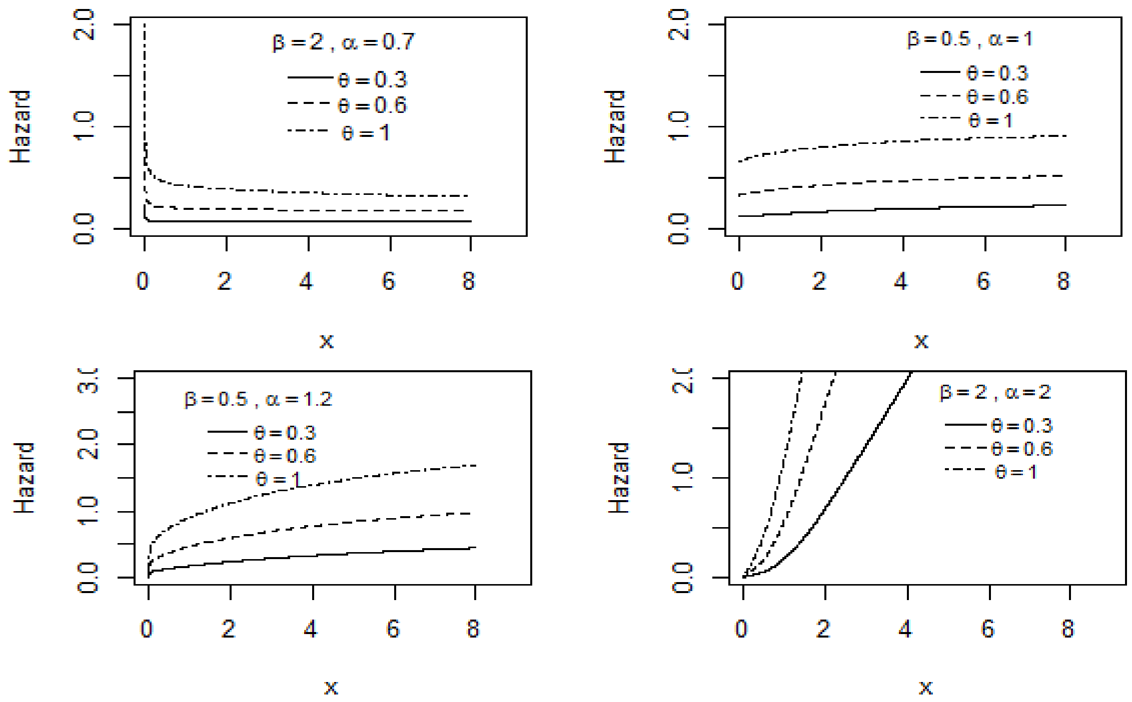

Fig. 2. Plots of the hazard rate function of the EPL distribution for different values of θ, β, and α.

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Moments, moment generating function, and associated measures

Theorem 1. Let X be a random variable that follows the EPL distribution with pdf as in (3), then the r^{th} row moment (about the origin) is given by

$$\mu_r' = E(x^r) = \frac{\alpha(\theta+\beta)+\beta r}{\alpha^2\theta^{\frac{r}{\alpha}}(\theta+\beta)} r\Gamma\frac{r}{\alpha}, \qquad (7)$$

and the moment generating function (mgf) is given by

$$M_X(t) = \sum_{n=1}^{\infty}\frac{t^n}{(n-1)!}\frac{[\alpha(\theta+\beta)+n\beta]}{\alpha^2\theta^{\frac{n}{\alpha}}(\theta+\beta)}\Gamma\frac{n}{\alpha}, \qquad (8)$$

where $\Gamma a = \int_0^\infty x^{a-1}e^{-x}dx$.

Proof: $\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx$.

For $X \sim EPL(\theta,\beta,\alpha)$, we have

$$\mu_r' = \frac{\alpha\theta^2}{\theta+\beta}\int_0^\infty x^r(1+\beta x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}dx$$
$$= \frac{\alpha\theta^2}{\theta+\beta}\left[\int_0^\infty x^{r+\alpha-1}e^{-\theta x^\alpha}dx + \beta\int_0^\infty x^{r+2\alpha-1}e^{-\theta x^\alpha}dx\right].$$

Letting $y = \theta x^\alpha$, we have

$$\mu_r' = \frac{\alpha\theta^2}{\theta+\beta}\left[\int_0^\infty \frac{y^{\frac{r}{\alpha}}e^{-y}}{\alpha\theta^{\frac{r}{\alpha}+1}}dy + \beta\int_0^\infty \frac{y^{\frac{r}{\alpha}+1}e^{-y}}{\alpha\theta^{\frac{r}{\alpha}+2}}dy\right].$$

Using the $\int_0^\infty x^{a-1}e^{-x}dx = \Gamma a = (a-1)\Gamma(a-1), a>0$, the above expression is reduced to

$$\mu_r' = \frac{\mathrm{r}[\alpha(\theta+\beta)+\beta r]}{\alpha^2\theta^{\frac{r}{\alpha}}(\theta+\beta)}\Gamma\frac{r}{\alpha},$$

The mgf of a continuous random variable X, when it exists, is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}f(x)dx.$$

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For $X \sim EPL(\theta, \beta, \alpha)$, we have

$$M_X(t) = \frac{\alpha\theta^2}{\theta+\beta}\int_0^\infty e^{tx}(1+\beta x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}dx.$$

Using the series expansion, $e^{tx} = \sum_{n=0}^{\infty}\frac{t^n x^n}{n!}$, the above expression is reduced to

$$M_X(t) = \frac{\alpha\theta^2}{\theta+\beta}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\int_0^\infty x^{n+\alpha-1}e^{-\theta x^\alpha}dx + \beta\int_0^\infty x^{n+2\alpha-1}e^{-\theta x^\alpha}dx\right].$$

Using the same ideas above, we end up with

$$M_X(t) = \sum_{n=1}^{\infty}\frac{t^n}{(n-1)!}\frac{[\alpha(\theta+\beta)+n\beta]}{\alpha^2\theta^{\frac{n}{\alpha}}(\theta+\beta)}\Gamma\frac{n}{\alpha},$$

where $\Gamma a = \int_0^\infty x^{a-1}e^{-x}dx$.

Therefore, the mean and the variance of the EPL distribution, respectively, are

$$\mu = \mu_1' = \frac{[\alpha(\theta+\beta)+\beta]}{\alpha^2\theta^{\frac{1}{\alpha}}(\theta+\beta)}\Gamma\frac{1}{\alpha}, \text{ and}$$

$$\sigma^2 = \mu_2' - \mu^2 = \frac{1}{\alpha^4\theta^{\frac{2}{\alpha}}(\theta+\beta)^2}\left[2\alpha^2(\theta+\beta)[\alpha(\theta+\beta)+2\beta]\Gamma\frac{2}{\alpha} - [\alpha(\theta+\beta)+\beta]^2\Gamma^2\frac{1}{\alpha}\right]$$

The *skewness* and *kurtosis* measures can be obtained from the expressions

$$skewness = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\sigma^3}$$

$$curtosis = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\sigma^4},$$

upon substituting for the row moments in (7).

Quantile function

Theorem 2. Let X be a random variable with pdf as in (3), the quantile function, say $Q(p)$ is

$$Q(p) = \left[-\frac{1}{\beta} - \frac{1}{\theta} - \frac{1}{\beta\theta}W_{-1}\left(-\frac{(\beta+\theta)(1-p)}{e^{(\beta+\theta)}}\right)\right]^{1/\alpha},$$

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where $\theta, \beta, \alpha > 0,\ p \in (0,1)$, and $W_{-1}(.)$ is the negative Lambert W function.

Proof: We have $Q(p) = F^{-1}(p),\ p \in (0,1)$ which implies $F(Q(p)) = p$, by substitution we get

$[\theta + \beta + \beta\theta(Q(p))^{\alpha}]\mathrm{e}^{-\theta(Q(p))^{\alpha}} = (\theta + \beta)(1 - p)$, multiply both sides by

$-e^{-(\theta+\beta)}$ and rise them to β, we have the Lambert equation,

$-[\theta + \beta + \beta\theta(Q(p))^{\alpha}]\mathrm{e}^{-\left(\theta+\beta+\beta\theta(Q(p))^{\alpha}\right)} = -(\theta + \beta)(1 - p)e^{-(\theta+\beta)}$. Hence we have the negative Lambert W function of the real argument $-(\theta + \beta)(1 - p)e^{-(\theta+\beta)}$ i.e.,

$$W_{-1}((\theta + \beta)(1 - p)e^{-(\theta+\beta)}) = -[\theta + \beta + \beta\theta(Q(p))^{\alpha}]$$

solving this equation for $Q(P)$, the proof is complete.

Special cases of the EIL distribution

The EPL distribution contains some well-known distributions as sub-models, described below in brief some examples.

Lindley distribution

The original Lindley distribution shown by Lindley (1958) is a special case of the EPL distribution; $\alpha = \beta = 1$. Using (3) and (4), the pdf and cdf are given by

$$\mathrm{f}(x;\theta) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}; \theta, x > 0 \text{ and } \mathrm{F}(x;\theta) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}; \theta, x > 0.$$

The associated hazard rate function using (6) is given by $\mathrm{h}(x) = \frac{\theta^2(1 + x)}{\theta + 1 + \theta x}, x > 0$. Using (7), the r^{th} row moment (about the origin) is given by $\mu_r' = \frac{(\theta + r + 1)\,\mathrm{r}!}{\theta^r(\theta + 1)}$. Substitute in the general form of mgf in (8), we have the Lindley mgf, $M_X(t) = \sum_{n=1}^{\infty} t^n \frac{\theta + n + 1}{\theta^n(\theta + 1)}$. The mean and the variance of Lindley distribution are then given, respectively, by

$$\mu = \frac{(\theta + 2)}{\theta(\theta + 1)} \text{ and } \sigma^2 = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta + 1)^2}.$$

Two-parameter Lindley distribution

The two-parameter Lindley distribution proposed by Shanker et al. (2013) is a special case of the EPL distribution; $\alpha = 1$. Using (3) and (4), the pdf and cdf are given by

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$\mathrm{f}(x;\theta,\beta)=\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x};\theta,\beta,x>0$ and $\mathrm{F}(x;\theta,\beta)=1-\frac{\theta+\beta+\theta\beta x}{\theta+\beta}e^{-\theta x};\theta,\beta,x>0.$

The associated hazard rate function using (6) is given by $\mathrm{h}(x)=\frac{\theta^2(1+\beta x)}{\theta+\beta+\beta\theta x};x>0.$ Using (7), the r^{th} row moment (about the origin) is given by $\mu_r'=\frac{(\theta+\beta+r\beta)\mathrm{r}!}{\theta^r(\theta+\beta)}$. Substitute in the general form of mgf in (8), we have the Lindley mgf, $M_X(t)=\sum_{n=1}^{\infty}t^n\frac{\theta+\beta+n\beta}{\theta^n(\theta+\beta)}$. The mean and the variance of Lindley distribution are then given, respectively, by

$$\mu=\frac{(\theta+2\beta)}{\theta(\theta+\beta)}\text{ and }\sigma^2=\frac{\theta^2+4\theta\beta+2\beta^2}{\theta^2(\theta+\beta)^2}.$$

Power Lindley distribution

The two-parameter Lindley distribution proposed by Ghitany et al. (2013 is a special case of the EPL distribution; $\beta=1$. Using (3) and (4), the pdf and cdf are given by

$$\mathrm{f}(x;\theta,\alpha)=\frac{\alpha\theta^2}{\theta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}};\theta,\alpha,x>0\text{ and}$$
$$\mathrm{F}(x;\theta,\alpha)=1-\frac{\theta+1+\theta x^{\alpha}}{\theta+1}e^{-\theta x^{\alpha}};\theta,\alpha,x>0.$$

The associated hazard rate function using (6) is given by $\mathrm{h}(x)=\frac{\alpha\theta^2x^{\alpha-1}(1+x^{\alpha})}{\theta+1+\theta x^{\alpha}};x>0.$ Using (7), the r^{th} row moment (about the origin) is given by $\mu_r'=\frac{\alpha(\theta+1)+r)}{\alpha^2\theta^{\frac{r}{\alpha}}(\theta+1)}r\Gamma\frac{r}{\alpha}$. Substitute in the general form of mgf in (8), we have the Lindley mgf, $M_X(t)=\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!}\frac{[\alpha(\theta+1)+n]}{\alpha^2\theta^{\frac{n}{\alpha}}(\theta+1)}\Gamma\frac{n}{\alpha}$. The mean and the variance of Lindley distribution are then given, respectively, by

$$\mu=\frac{\alpha\theta+\alpha+1)}{\alpha^2\theta^{\frac{1}{\alpha}}(\theta+1)}\Gamma\frac{1}{\alpha}\text{ and}$$
$$\sigma^2=\frac{1}{\alpha^4\theta^{\frac{2}{\alpha}}(\theta+1)^2}\left[2\alpha^2(\theta+1)(\alpha\theta+\alpha+2)\Gamma\frac{2}{\alpha}-(\alpha\theta+\alpha+1)^2\Gamma^2\frac{1}{\alpha}\right].$$

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Weibull distribution

A two-parameter Weibull distribution is a special case of the EPL distribution; $\beta = 0$. Using (3) and (4), the pdf and cdf are given by

$$\mathrm{f}(x;\theta,\alpha) = \alpha\theta x^{\alpha-1}e^{-\theta x^{\alpha}};\theta,\alpha > 0 \text{ and } \mathrm{F}(x;\theta,\alpha) = 1 - e^{-\theta x^{\alpha}};\theta,\alpha > 0.$$

The associated hazard rate function using (6) is given by $\mathrm{h}(x) = \alpha\theta x^{\alpha-1}; x > 0$. Using (7), the r^{th} row moment (about the origin) is given by $\mu_r' = \dfrac{r}{\alpha\theta^{\frac{r}{\alpha}}}\Gamma\dfrac{r}{\alpha}$. Substitute in the general form of mgf in (8), we have the Lindley mgf, $M_X(t) = \sum_{n=1}^{\infty}\dfrac{t^n}{(n-1)!}\dfrac{1}{\alpha\theta^{\frac{n}{\alpha}}}\Gamma\dfrac{n}{\alpha}$. The mean and the variance of Lindley distribution are then given, respectively, by

$$\mu = \frac{1}{\alpha\theta^{\frac{1}{\alpha}}}\Gamma\frac{1}{\alpha} \text{ and } \sigma^2 = \frac{2\alpha\Gamma\frac{2}{\alpha} - \Gamma^2\frac{1}{\alpha}}{\alpha^2\theta^{\frac{2}{\alpha}}}.$$

Stochastic orderings

Stochastic orderings of positive continuous random variables is an important tool to judge the comparative behavior. A random variable X is said to be smaller than a random variable Y in the following contests:

(i) Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \leq F_Y(x)\ \forall x$;

(ii) Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)\ \forall x$;

(iii) Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)\ \forall x$;

(iv) Likelihood ratio order $(X \leq_{lr} Y)$ if $f_X(x)/f_Y(x)$ decreases in x.

The following implications (Shaked and Shanthikumar, 1994) are well known that

$$\begin{array}{ccccc} X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y & \Rightarrow & X \leq_{mrl} Y \\ & & \Downarrow & & \\ & & X \leq_{st} Y & & \end{array}$$

The following theorem shows that the EPL distribution is ordered with respect to "likelihood ratio" ordering.

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Theorem 3. Let $X \sim PL(\theta_1, \beta_1, \alpha_1)$ and $Y \sim PL(\theta_2, \beta_2, \alpha_2)$. If $\beta_1 = \beta_2$ and $\theta_1 \ge \theta_2$ (or if $\theta_1 = \theta_2$ and $\beta_1 \ge \beta_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: for $\alpha_1 = \alpha_2 = \alpha$ we have

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{\theta_1}{\theta_2}\right)^2 \left(\frac{\theta_2+\beta_2}{\theta_1+\beta_1}\right)\left(\frac{1+\beta_1 x^\alpha}{1+\beta_2 x^\alpha}\right) e^{-(\theta_1-\theta_2)x^\alpha}; \alpha, x > 0 ,$$

and

$$\log\frac{f_X(x)}{f_Y(x)} = 2\log\left(\frac{\theta_1}{\theta_2}\right) + \log\left(\frac{\theta_2+\beta_2}{\theta_1+\beta_1}\right) + \log(1+\beta_1 x^\alpha) - \log(1+\beta_2 x^\alpha) - (\theta_1-\theta_2)x^\alpha .$$

Thus

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{\alpha\beta_1 x^{\alpha-1}}{1+\beta_1 x^\alpha} - \frac{\alpha\beta_2 x^{\alpha-1}}{1+\beta_2 x^\alpha} - \alpha(\theta_1-\theta_2)x^{\alpha-1}$$
$$= \left(\frac{\beta_2-\beta_1}{(1+\beta_1 x^\alpha)(1+\beta_2 x^\alpha)} + (\theta_2-\theta_1)\right)\alpha x^{\alpha-1}$$

Case (i): If $\beta_1 = \beta_2$ and $\theta_1 \ge \theta_2$, then

$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \le_{lr} Y$ and hence

$X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Case (ii): If $\beta_1 \ge \beta_2$ and $\theta_1 = \theta_2$, then

$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Estimation and inference

Let $X_1,\ldots,X_n$ be a random sample, with observed values $x_1,\ldots,x_n$ from $EPL(\theta,\beta,\alpha)$ distribution. Let $\Theta = (\theta,\beta,\alpha)$ be the 3×1 parameter vector. The log likelihood function is given by

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$$\ln = n[\log\alpha + 2\log\theta - \log(\theta+\beta)] + \sum_{i=1}^{n}\log(1+\beta x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n}x_i^{\alpha}.$$

Then the score function is given by

, $U_n(\Theta) = (\partial\ln/\partial\theta, \partial\ln/\partial\beta, \partial\ln/\partial\alpha)^T$ are

$$\frac{\partial\ln}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+\beta} - \sum_{i=1}^{n}x_i^{\alpha},$$
$$\frac{\partial\ln}{\partial\beta} = \frac{-n}{\theta+\beta} + \sum_{i=1}^{n}\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}},$$
$$\frac{\partial\ln}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\beta x_i^{\alpha}\log x_i}{1+\beta x_i^{\alpha}} + \sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n}x_i^{\alpha}\log x_i.$$

The maximum likelihood estimation (MLE) of Θ say Θ is obtained by solving the nonlinear system $U_n(\mathrm{x};\Theta) = 0$. This nonlinear system of equations does not have a closed form. For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix

$$I_n(\Theta) = -\begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} & I_{\theta\alpha} \\ I_{\beta\theta} & I_{\beta\beta} & I_{\beta\alpha} \\ I_{\alpha\theta} & I_{\alpha\beta} & I_{\alpha\alpha} \end{bmatrix}$$

where the elements of $I_n(\Theta)$ are the second partial derivatives of $U_n(\Theta)$. Under standard regular conditions for large sample approximation (Cox and Hinkley, 1974) that fulfilled for the proposed model, the distribution of Θ is approximately $N_3(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta) = \mathrm{E}[\mathrm{I}_n(\Theta)]$. Whenever the parameters are in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\Theta - \Theta)$ is $N_3(0, J(\Theta)^{-1})$, where $J(\Theta)^{-1} = \lim_{n\to\infty} n^{-1} I_n(\Theta)$ is the unit information matrix and p is the number of parameters of the distribution. The asymptotic multivariate normal $N_3(\Theta, I_n(\Theta)^{-1})$ distribution of Θ can be used to approximate the confidence interval for the parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for parameter Θ_i is given by

$$(\Theta_i - Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}, \Theta_i + Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}),$$

where I^{ii} is the (i,i) diagonal element of $I_n(\Theta)^{-1}$ for $i = 1,\ldots,3$ and $Z_{\frac{\gamma}{2}}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

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Rényi entropy

Entropy is a measure of variation of the uncertainty in the distribution of any random variable. It provides important tools to indicate variety in distributions at particular moments in time and to analyze evolutionary processes over time. For a given probability distribution, Rényi (1961) gave an expression of the entropy function, so-called Rényi entropy, defined by

$$\mathrm{Re}(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\}$$

where $\gamma>0$ and $\gamma\neq 0$. For the EPL distribution in (3), we have

$$\mathrm{Re}(\gamma)=\frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha\theta^2}{\theta+\beta}\right)^{\gamma}\int_0^{\infty}\left(1+\beta x^{\alpha}\right)^{\gamma}x^{\gamma(\alpha-1)}e^{-\theta\gamma x^{\alpha}}dx\right\}.$$

Now using the fact $(1+z)^{\gamma}=\sum_{j=0}^{\infty}\binom{\gamma}{j}z^{j}$, we have

$$\mathrm{Re}(\gamma)=\frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha\theta^2}{\theta+\beta}\right)^{\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\beta^{j}\int_0^{\infty}x^{j\alpha+\gamma(\alpha-1)}e^{-\theta\gamma x^{\alpha}}dx\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\frac{\alpha^{\gamma-1}}{(\theta+\beta)^{\gamma}}\frac{\theta^{\gamma(1+1/\alpha)-1/\alpha}}{\gamma^{\gamma(1-1/\alpha)+1/\alpha}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\left(\frac{\beta}{\theta\gamma}\right)^{j}\Gamma(\gamma+j-\gamma/\alpha+1)\right\}$$

Applications

In this section, we demonstrate the applicability of the EIL model for real data. Bader and Priest (1982) obtained tensile strength measurements on 1000-carbon fiber impregnated tows at four different gauge lengths. These carbon fiber micro-composite specimens were tested under tensile load until breakage, and the breaking stress was recorded (in gigapascals, GPa). At the gauge length of 50 mm, n ¼ 30 observed breaking stresses were recorded. The data are listed in Table 1. The data were recently used as an illustrative example for power Lindley distribution by Ghitany et al. (2013).

Table 1: Carbon fibers tensile strength

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966
1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240
2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434
2.435	2.478	2.490	2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629
2.633	2.642	2.648	2.684	2.697	2.726	2.770	2.773	2.800	2.809	2.818
2.821	2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096	3.128	3.233
3.433	3.585	3.585								

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For this data, we fit the proposed EPL, $EPL(\theta,\beta,\alpha)$, as well as the sub models that were introduced in Section 6.

The expectation–maximization (EM) algorithm is used to estimate the model parameters. The MLEs of the parameters, the Kolmogorov–Smirnov statistics (K-S) with its respective p-value, the maximized log likelihood for the above distributions, as well as our proposed model are given in Table 2. They indicate that the EPL distribution (proposed model) fits the data better than the other distributions. The $EPL(\theta,\beta,\alpha)$ takes the smallest K-S test statistic value and the largest value of its corresponding p-value. In addition, it takes the largest log likelihood. The fitted densities and the empirical distribution versus the fitted cumulative distributions of all models for this data are shown in Figs. 3 and 4, respectively.

Table 2: Parameter estimates, K-S statistic, p-value, and logL of the carbon fibers tensile strength

Dist.	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	K-S	p-value	$\log L$
$EPL(\theta,\beta,\alpha)$	0.0584	98.9	3.7313	0.0429	0.9996	-48.9
$EPL(\theta,1,\alpha)$	0.0450	-	3.8678	0.0442	0.9993	-49.06
$EPL(\theta,0,\alpha)$	0.0100	-	4.8175	0.1021	0.4685	-50.65
$EPL(\theta,\beta,1)$	0.8158	4504.4	-	0.3614	0.000	-105.7
$EPL(\theta,1,1)$	0.6545	-	-	0.4011	0.000	-119.2

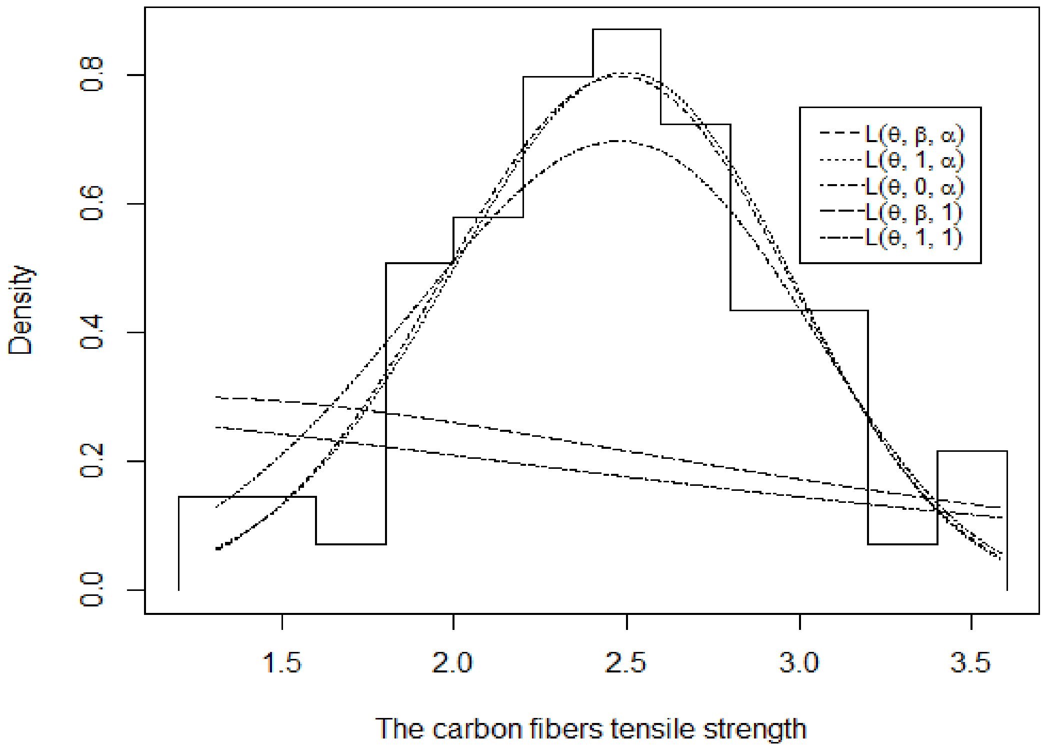

Fig. 3. Plot of the fitted densities of the models in Table 2.

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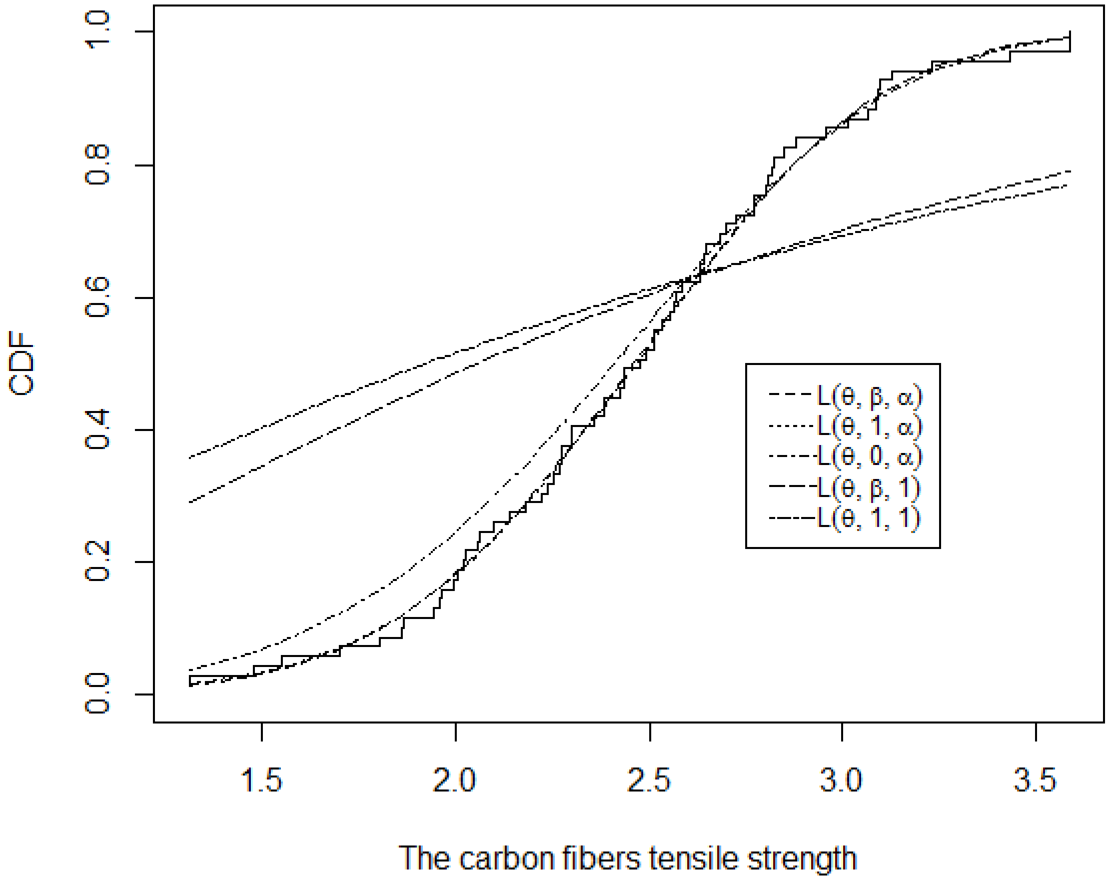

Fig. 4. Plot of the fitted CDFs of the models in Table 2.

CONCLUDING REMARKS

In this paper, a new three-parameter distribution called "the extended power Lindley" distribution is introduced and studied in detail. This model has more flexibility than other types of Lindley distributions (one and two parameters) due to the shape of its density as well as its hazard rate functions. The density of the new distribution can be expressed as a two-component Weibull density functions and a generalized gamma density function. Maximum likelihood estimates of the parameters and its asymptotic confidence intervals for model parameters are shown. Application of the proposed distribution to real data shows better fit than many other well-known distributions, such as Lindley, two-parameter Lindley, Weibull, and power Lindley.

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