European Journal of Statistics and Probability
Vol.3, No.2, pp.24-37, June 2015
Published by European Centre for Research Training and Development UK (www.eajournals.org)

ESTIMATION OF PARAMETERS OF THE MARSHALL-OLKIN WEIBULL DISTRIBUTION FROM PROGRESSIVELY CENSORED SAMPLES

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ABSTRACT: *This model can be considered as another useful 3-parameter generalization of the Weibull distribution. The estimation of parameters of Marshall and Olkin Weibull distribution under progressive censoring is investigated, maximum likelihood estimators of the unknown parameters are obtained using statistical software (Mathematica), MLE performs for different sampling schemes with different sample sizes is observed, and the asymptotic variance covariance matrix is computed also.*

KEYWORDS Modified Weibull distribution, progressive type II censoring, Maximum likelihood estimators, Asymptotic variance covariance matrix.

INTRODUCTION

Marshall and Olkin (1997) proposed a modification of the standard Weibull model through the introduction of an additional parameter $\alpha\ (0<\alpha<\infty)$, leading to a cumulative distribution function of the form

$$F(t)=\left(1-\frac{\alpha e^{-(\lambda t)^{\beta}}}{1-(1-\alpha)e^{-(\lambda t)^{\beta}}}\right),\qquad \alpha,\lambda,\beta>0. \tag{1-1}$$

With probability distribution function

$$f(t)=\frac{(\lambda\alpha\beta)(\lambda t)^{\beta-1}e^{-(\lambda t)^{\beta}}}{\left(1-(1-\alpha)e^{-(\lambda t)^{\beta}}\right)^{2}}. \tag{1-2}$$

The quantile function is given by

$$t=F^{-1}(u)=\left(\frac{\log\left[\frac{1-u(1-\alpha)}{1-u}\right]}{\lambda}\right)^{\frac{1}{\beta}} \tag{1-3}$$

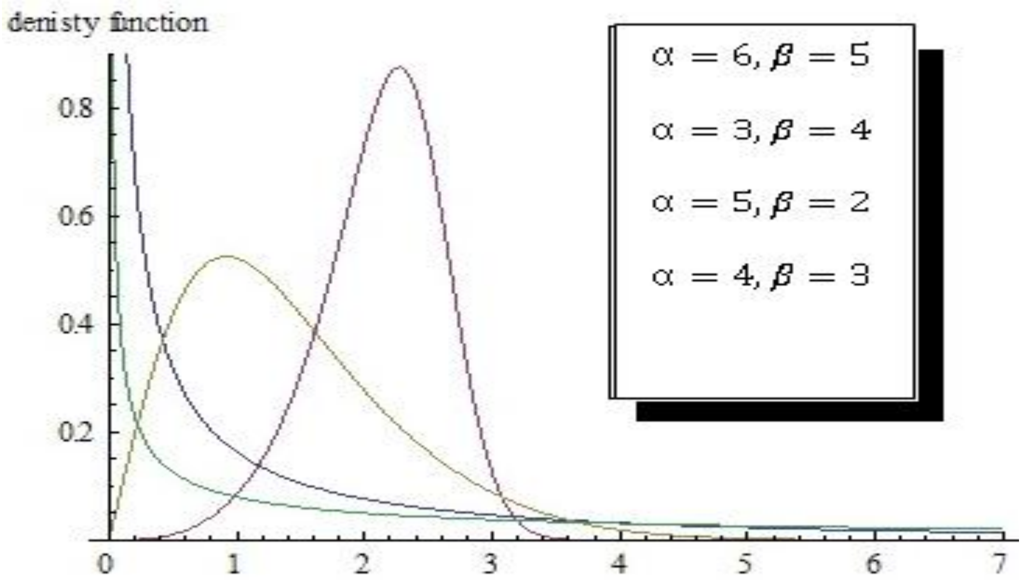
ISSN 2055-0154(Print), ISSN 2055-0162(Online)

The hazard rate is

$$h(t)=\frac{(\lambda\beta)(\lambda t)^{\beta-1}}{\left(1-(1-v)e^{-(\lambda t)^{\beta}}\right)}. \qquad (1\text{-}4)$$

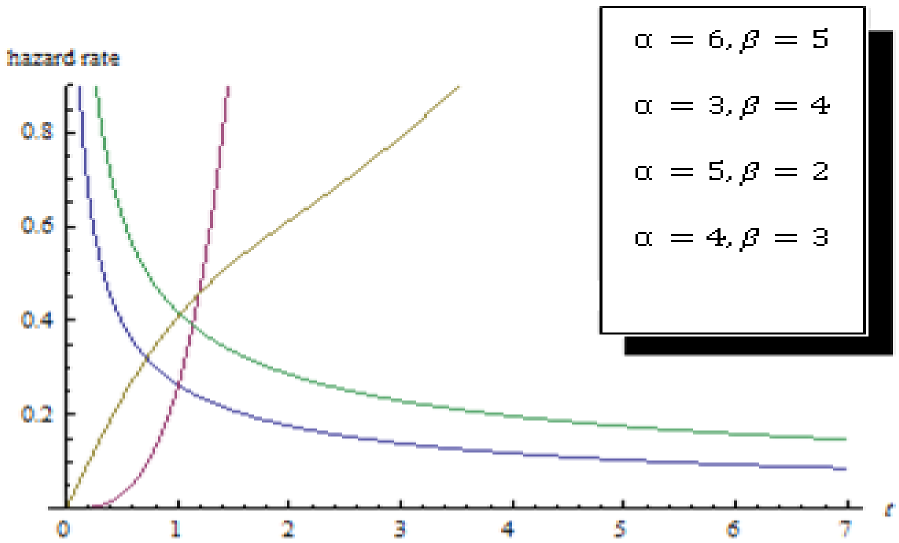
The following diagram provides graphs of the modified Weibull density function for selected values of the parameters.

Fig (1-1) density function of modified Weibull distribution $\lambda = 2$



The following diagram provides graphs of modified Weibull hazard rates

Fig (1-2) hazard rates of modified Weibull distribution $\lambda = 2$



ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Marshal and Olkin (1997) studied the hazard rate and found that it is increasing if $\alpha \geq 1$, $\beta \geq 1$ and decreasing if $\alpha \leq 1$, $\beta \leq 1$. If $\beta > 1$ then the hazard rate is initially increasing and eventually increasing , but there may be one interval where it is decreasing.Similarly , if $\beta < 1$ then the hazard rate is initially decreasing and eventually decreasing , but there may be one interval where it is increasing .

Marshal and Olkin (1997) proposed a way of introducing a parameter, to expand a family of distributions. Ramesh C. Gupta et al. (2010) compare the modified distribution and the original distribution with respect to some stochastic orderings. Also investigated thoroughly the monotonicity of the failure rate of the resulting distribution when the baseline distribution is taken as weibull.

It turns out that the failure rate is increasing, decreasing, or non-monotonic with one or two turning points depending on the parameters. For non-monotonic types, the turning points of the failure rate are estimated and their confidence intervals are provided. Simulation studies are carried out to examine the performance of these intervals.

MAXIMUM LIKELIHOOD ESTIMATION

In case complete data

Consider a random sample consisting of n observations. The likelihood function of this sample is

$$L(t_1,t_2,...............,t_n;\alpha,\beta,\lambda)=\prod_{i=1}^{n} f(t_i) \qquad (1-5)$$

$$L(t,\alpha,\lambda,\beta)=\frac{(\alpha\lambda\beta)^n \prod_{i=1}^{n}(\lambda t_i)^{\beta-1} e^{-\prod_{i=1}^{n}(\lambda t_i)^{\beta}}}{\prod_{i=1}^{n}(1-(1-\alpha)e^{-(\lambda t_i)^{\beta}})^2} \qquad (1-6)$$

On taking logarithms of (1-6),

$$L.L = n\log(\alpha\lambda\beta)-\sum_{i=1}^{n}(\lambda t_i)^{\beta}+(\beta-1)\sum_{i=1}^{n}(\lambda t_i)-2\sum_{i=1}^{n}\log(1-(1-\alpha)e^{-(\lambda t)^{\beta}}) \qquad (1-7)$$

Differentiating (1-7) with respect to α, λ and β and equating to zero are:

$$\frac{\partial L.L}{\partial \alpha}=\frac{n}{\alpha}-2\sum_{i=1}^{n}\frac{e^{-(\lambda t_i)^{\beta}}}{1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}}=0, \qquad (1-8)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\frac{\partial L.L}{\partial \lambda}=\frac{n}{\lambda}-\frac{\beta}{\lambda}*\sum_{i=1}^{n}(\lambda t_i)^{\beta}+\frac{(\beta-1)}{\lambda}-\frac{2\beta}{\lambda}\sum_{i=1}^{n}\frac{(1-\alpha)e^{-(\lambda t)^{\beta}}(\lambda t_i)^{\beta}}{(1-(1-\alpha)e^{-(\lambda t)^{\beta}})}=0, \qquad (1-9)$$

$$\frac{\partial L.L}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}(\lambda t_i)^{\beta}\log(\lambda t_i)+\sum_{i=1}^{n}\log(\lambda t_i)-2\sum_{i=1}^{n}\frac{(1-\alpha)e^{-(\lambda t)^{\beta}}(\lambda t_i)^{\beta}\log(\lambda t_i)}{1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}}=0, \qquad (1-10)$$

The maximum likelihood estimate (MLE) $(\hat{\alpha},\hat{\lambda},\hat{\beta})$ of (α,λ,β) is obtained by solving the nonlinear likelihood equations (1-8), (1-9) and (1-10). These equations cannot be solved analytically and popular software can be used to solve these equations numerically.

In case progressive type two right censored data.

Let $t_1, t_2, \ldots\ldots\ldots t_n$ be independent and identically distributed random lifetimes of n items. A type–ll progressively right censored sample may be obtained in the following way: the failure time of the first t_1 ,r_1 are not observed; at the time of the (r+1)st failure, denoted with t_{r+1},R_{r+1} number of the remaining units are withdrawn from the test randomly. At the time of second failure, denoted with t_2, R_{r+2}surviving items are removed at random from the remaining items, and so on. At the time of the mth failure, when m is a predetermined number all the remaining R_{r+m} items are censored.

Therefore, a progressively type-ll right censoring scheme is specified by integer numbers n, m and r_1 ,………r_{m-1} with the constraints

$n- m- r_1 -\ldots r_{m-1}\geq 0$, $n\geq m\geq 1$.

Let The likelihood function to be maximized when general progressive type II censored sample based on n independent with *F (t)* is *L (θ)*

$$L(\theta)=c^{*}\left[F(t_{r+1},\theta)\right]^{r}\prod_{i=r+1}^{m+r}f(t_i,\theta)\left[1-F(t_i,\theta)\right]^{R_i} \qquad (1-11)$$

Where,

$$,t_{r+1}<t_{r+2}<\ldots<t_m,$$

$$c^{*}=\frac{n!}{r!(n-r-1)!}(n-R_{r+1}-r-1)\ldots(n-R_{r+1}-\ldots-R_{m-1}-m+1),$$

The likelihood function given in Eq. (1.11) can be expressed as:

$$L(t;\alpha,\lambda,\beta)=c^{*}\left(1-\frac{\alpha e^{-(\lambda t_{r+1})^{\beta}}}{1-(1-\alpha)e^{-(\lambda t_{r+1})^{\beta}}}\right)^{r}\prod_{i=1}^{m+r}\frac{(\alpha\lambda\beta)(\lambda t_i)^{\beta-1}e^{-(\lambda t_i)^{\beta}}}{\left(1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}\right)^{2}}$$

$$\times\left(\frac{\alpha e^{-(\lambda t_i)^{\beta}}}{1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}}\right)^{R_i} \qquad (1-12)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

The log-likelihood function, denoted by $l\left(t;\alpha,\lambda,\beta\right)$, takes the form

$$l\left(t;\alpha,\lambda,\beta\right)=\text{constant}+\left(m+r\right)\log\left[\alpha\lambda\beta\right]+r\log\left[1-\frac{\alpha e^{-(\lambda t_{r+1})^{\beta}}}{1-(1-\alpha)e^{-(\lambda t_{r+1})^{\beta}}}\right]+$$

$$+\left(\beta-1\right)\sum_{i=1}^{m+r}\log\left[\lambda t_i\right]-\sum_{i=1}^{m+r}\left(\lambda t_i\right)^{\beta}-2\sum_{i=1}^{m+r}\log\left[1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}\right]+$$

$$+\sum_{i=1}^{m+r}R_i\log\left[\frac{\alpha e^{-(\lambda t_i)^{\beta}}}{1-(1-\alpha)e^{-(\lambda t_i)^{\beta}}}\right]. \qquad (1.13)$$

The three equations obtained by differentiating (1.13) with respect to α, λ and β and equating to zero are:

$$\frac{\partial\ell(t)}{\partial\alpha}=\frac{m+r}{\alpha}+\frac{r\left(-\frac{D(t_{r+1})}{(1-B(t_{r+1}))}+\frac{F(t_{r+1})}{(1-B(t_{r+1}))^2}\right.}{1-\frac{A(t_{r+1})}{1-B(t_{r+1})}}+2\sum_{i=r+1}^{m+r}\frac{D(t_i)}{1-B(t_i)}$$

$$+\sum_{i=r+1}^{m+r}\frac{D(t_i)(1-B(t_i))\left(\frac{D(t_{r+1})}{(1-B(t_{r+1}))}-\frac{F(t_{r+1})}{(1-B(t_{r+1}))^2}\right)R_i}{\alpha}=0, \qquad (1-14)$$

$$\frac{\partial\ell(t)}{\partial\lambda}=\frac{m+r}{\lambda}+\frac{(m-r)(\beta-1)}{\lambda}+\frac{r\left(\frac{A(t_{r+1})G(t_{r+1})}{1-B(t_{r+1})}+\frac{F(t_{r+1})G(t_{r+1})}{(1-B(t_{r+1}))^2}\right)}{1-\frac{A(t_{r+1})}{1-B(t_{r+1})}}-\sum_{i=r+1}^{m+r}G(t_i)-2\sum_{i=r+1}^{m+r}\frac{B(t_i)G(t_i)}{1-B(t_i)}$$

$$+\sum_{i=r+1}^{m+r}\frac{D(t_i)(1-B(t_i))R_i\left(-\frac{A(t_i)G(t_i)}{1-B(t_i)}-\frac{F(t_i)G(t_i)}{(1-B(t_i))^2}\right.}{\alpha}=0, \qquad (1-15)$$

$$\frac{\partial\ell(t)}{\partial\beta}=\frac{m+r}{\beta}+\frac{r\,\frac{(A(t_{r+1})C(t_{r+1})}{1-B(t_{r+1})}+\frac{F(t_{r+1})\,\alpha\,C(t_{r+1})}{(1-B(t_{r+1}))^2}}{1-\frac{A(t_{r+1})}{1-B(t_{r+1})}}+\sum_{i=r+1}^{m+r}\log[\lambda t_i]-\sum_{i=r+1}^{m+r}C(t_i)-2\sum_{i=r+1}^{m+r}\frac{B(t_i)C(t_i)}{1-B(t_i)}$$

$$\sum_{i=r+1}^{m+r}\frac{D(t_i)(1-B(t_i))R_i\left(-\frac{A(t_i)C(t_i)}{1-B(t_i)}-\frac{F(t_i)\,\alpha\,C(t_i)}{(1-B(t_i))^2}\right.}{\alpha}=0, \qquad (1-16)$$

Where,

$$A(t_{r+1}) = \alpha e^{-(\lambda t_{r+1})^{\beta}}$$
$$B(t_{r+1}) = (1-\alpha) e^{-(\lambda t_{r+1})^{\beta}}$$
$$C(t_{r+1}) = \log[\lambda t_{r+1}](\lambda t_{r+1})^{\beta}$$
$$D(t_{r+1}) = e^{-(\lambda t_{r+1})^{\beta}}$$
$$F(t_{r+1}) = (1-\alpha) e^{-2(\lambda t_{r+1})^{\beta}}$$
$$G(t_{r+1}) = \beta t_{r+1}\left(\lambda t_{r+1}\right)^{\beta-1}$$

The maximum likelihood estimate (MLE) $\left(\hat{\alpha},\hat{\lambda},\hat{\beta}\right)$ of $\left(\alpha,\lambda,\beta\right)$ is obtained by solving the nonlinear likelihood equations (1-14), (1-15) and (1-16). These equations cannot be solved analytically and popular software can be used to solve the equations numerically.

Fisher information matrix $I(\theta)$

$$I(\theta) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) \end{pmatrix}.$$

The elements of the sample information matrix, for progressively type II censored will be

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{m+r}{\alpha^2} - \frac{r\left(\frac{D(t_{r+1})}{(1-B(t_{r+1}))} + \frac{K(t_{r+1})}{(1-B(t_{r+1}))^2}\right)^2}{\left(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}\right)^2} + \frac{r\left(\frac{2D(t_{r+1})}{(1-B(t_{r+1}))^2} - \frac{2H(t_{r+1})}{(1-B(t_{r+1}))^3}\right)}{1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}}$$
$$-2\sum_{i=r+1}^{m+r} - \frac{Q(t_{r+1})}{(1-B(t_{r+1}))^2} + \sum_{i=r+1}^{m} \frac{J(t_{r+1})((1-B(t_i))(-\frac{2Q(t_i)}{(1-B(t_i))^2} + \frac{2H(t_i)}{(1-B(t_i))^3})R_i}{\alpha}$$
$$-\frac{J(t_{r+1})((1-B(t_i))(\frac{D(t_i)}{(1-B(t_i))} - \frac{K(t_i)}{(1-B(t_i))^2})R_i}{\alpha^2} + \frac{(\frac{D(t_i)}{(1-B(t_i))} - \frac{K(t_i)}{(1-B(t_i))^2})R_i}{\alpha} \qquad (1-17)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\lambda}=\frac{r\,(\frac{D(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))}+\frac{L(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2K(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2S(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^3})}{1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}}$$

$$-\frac{r(-\frac{D(t_{r+1})}{(1-B(t_{r+1}))}+\frac{K(t_{r+1})}{(1-B(t_{r+1}))^2})(\frac{A(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^2})}{(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))})^2}$$

$$-2\sum_{i=r+1}^{m+r}(\frac{D(t_i)\,G(t_i)}{(1-B(t_i))}-\frac{L(t_i)\,G(t_i)}{(1-B(t_i))^2})$$

$$+\sum_{i=r+1}^{m+r}(\frac{J(t_i)\,(1-B(t_i))\,(\frac{D(t_i)}{(1-B(t_i))}-\frac{K(t_i)}{(1-B(t_i))^2})\,G(t_i)\,R_i}{\alpha}$$

$$+\frac{(1-\alpha)(\frac{D(t_i)}{(1-B(t_i))}-\frac{K(t_i)}{(1-B(t_i))^2})\,G(t_i)\,R_i}{\alpha}$$

$$+\frac{J(t_i)\,((1-B(t_i))\,R_i\,(-\frac{D(t_i)\,G(t_i)}{(1-B(t_i))}-\frac{L(t_i)\,G(t_i)}{(1-B(t_i))^2}+\frac{2\,K(t_i)\,G(t_i)}{(1-B(t_i))^2}+\frac{2\,S(t_{r+1})\,G(t_{r+1})}{(1-B(t_i))^3})}{\alpha}\qquad(1-18)$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\beta}=\frac{r\,(\frac{D(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))}+\frac{L(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2\,K(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2\,S(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))^3})}{1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}}$$

$$-\frac{r\,(\frac{D(t_{r+1})}{(1-B(t_{r+1}))}+\frac{K(t_{r+1})}{(1-B(t_{r+1}))^2})\,(\frac{A(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))^2})}{(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))})^2}$$

$$-2\sum_{i=r+1}^{m+r}(-\frac{D(t_i)\,C(t_i)}{(1-B(t_i))}-\frac{L(t_i)\,C(t_i)}{(1-B(t_i))^2})$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$+\sum_{i=1}^{m+r}\frac{J(t_i)\,(1-B(t_i))\,\left(\frac{D(t_i)}{(1-B(t_i))}-\frac{K(t_i)}{(1-B(t_i))}\right)C(t_i)\,R_i}{\alpha}$$

$$+\frac{(1-\alpha)\,\left(\frac{D(t_i)}{(1-B(t_i))}-\frac{K(t_i)}{(1-B(t_i))^2}\right)C(t_i)\,R_i}{\alpha}$$

$$\frac{J(t_i)\,((1-B(t_i))\,R_i\,\left(-\frac{D(t_i)\,C(t_i)}{(1-B(t_i))}-\frac{L(t_i)\,C(t_i)}{(1-B(t_i))^2}+\frac{2\,K(t_i)\,C(t_i)}{(1-B(t_i))^2}+\frac{2\,S(t_i)\,C(t_i)}{(1-B(t_i))^3}\right.}{\alpha} \qquad (1-19)$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{m+r}{\lambda^2}-\frac{(m-r)\,(\beta-1)}{\lambda^2}-\frac{r\left(\frac{A(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^2}\right)^2}{\left(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}\right)^2}$$

$$+\frac{r\left(\frac{A(t_{r+1})\,(\beta-1)\,N(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})\,(\beta-1)\,N(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{A(t_{r+1})\,w(t_{r+1})}{(1-B(t_{r+1}))}-\frac{3\,a(t_{r+1})\,w(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2\,d(t_{r+1})\,w(t_{r+1})}{(1-B(t_{r+1}))^3}\right)}{1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}}$$

$$-\sum_{i=r+1}^{m+r}(\beta-1)\,N(t_i)-2\sum_{i=r+1}^{m+r}\left(\frac{(1-B(t_i))\,(\beta-1)\,N(t_i)}{(1-B(t_i))}-\frac{(1-B(t_i))\,w(t_i)}{(1-B(t_i))}-\frac{f(t_i)\,N(t_i)}{(1-B(t_i))^2}\right)$$

$$+\frac{(1-\alpha)\,R_i\,G(t_i)\,\left(-\frac{A(t_i)\,G(t_i)}{(1-B(t_i))}-\frac{a(t_i)\,G(t_i)}{(1-B(t_i))^2}\right)}{\alpha}$$

$$+\frac{J(t_i)\,(1-B(t_i))\,R_i\,\left(-\frac{A(t_i)\,(\beta-1)\,N(t_i)}{(1-B(t_i))}-\frac{a(t_i)\,(\beta-1)\,N(t_i)}{(1-B(t_i))^2}+\frac{A(t_i)\,w(t_i)}{(1-B(t_i))}+\frac{3\,a(t_i)\,w(t_i)}{(1-B(t_i))^2}+\frac{2\,d(t_i)\,w(t_i)}{(1-B(t_i))^3}\right)}{\alpha}\Bigg)$$

$$(1-20)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\frac{\partial^2 \ell}{\partial\lambda\,\partial\beta} = \frac{m-r}{\lambda} - \frac{r\,\left(\frac{A(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))} + \frac{a(t_{r+1})\,G(t_{r+1})}{(1-B(t_{r+1}))^2}\right)\left(\frac{A(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))} + \frac{a(t_{r+1})\,C(t_{r+1})}{(1-B(t_{r+1}))^2}\right)}{\left(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}\right)^2}$$

$$+\frac{1}{1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))}}\,r\,\left(\frac{A(t_{r+1})\,g(t_{r+1})}{(1-B(t_{r+1}))} + \frac{a(t_{r+1})\,g(t_{r+1})}{(1-B(t_{r+1}))^2} + \frac{A(t_{r+1})\,b(t_{r+1})}{(1-B(t_{r+1}))}\right.$$

$$-2\sum_{i=r+1}^{m+r}\left(\frac{B(t_i)\,g(t_i)}{(1-B(t_i))} + \frac{B(t_i)\,b(t_i)}{(1-B(t_i))} - \frac{B(t_i)\,e(t_i)}{(1-B(t_i))} - \frac{f(t_i)\,e(t_i)}{(1-B(t_i))^2}\right)$$

$$+\sum_{i=r+1}^{m+r}\left(\frac{J(t_i)\,(1-B(t_i))\,R_i\,C(t_i)\,\left(-\frac{A(t_i)\,G(t_i)}{(1-B(t_i))} - \frac{a(t_i)\,G(t_i)}{(1-B(t_i))^2}\right)}{\alpha}\right.$$

$$-\frac{(1-\alpha)\,R_i\,C(t_i)\,\left(-\frac{A(t_i)\,G(t_i)}{(1-B(t_i))} - \frac{a(t_i)\,G(t_i)}{(1-B(t_i))^2}\right)}{\alpha}$$

$$+\frac{1}{\alpha}\,J(t_i)\,((1-B(t_i))\,R_i\,\left(-\frac{A(t_i)\,g(t_i)}{(1-B(t_i))} - \frac{a(t_i)\,g(t_i)}{(1-B(t_i))^2} - \frac{A(t_i)\,b(t_i)}{(1-B(t_i))} - \frac{a(t_i)\,b(t_i)}{(1-B(t_i))^2}\right.$$

$$+\frac{A(t_i)\,e(t_i)}{(1-B(t_i))} + \frac{3\,a(t_i)\,e(t_i)}{(1-B(t_i))^2} + \frac{2d\,(t_i)\,e(t_i)}{(1-B(t_i))^3}\Big)\Big) \qquad (1-21)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\frac{\partial^2\ell}{\partial\beta^2}=-\frac{m+r}{\beta^2}-\frac{r(\frac{A(t_{r+1})C(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})C(t_{r+1})}{(1-B(t_{r+1}))^2})^2}{(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))})^2}$$

$$+\frac{r(\frac{A(t_{r+1})c(t_{r+1})}{(1-B(t_{r+1}))}+\frac{a(t_{r+1})c(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{A(t_{r+1})h(t_{r+1})}{(1-B(t_{r+1}))}-\frac{3a(t_{r+1})h(t_{r+1})}{(1-B(t_{r+1}))^2}-\frac{2d(t_{r+1})h(t_{r+1})}{(1-B(t_{r+1}))^3})}{(1-\frac{A(t_{r+1})}{(1-B(t_{r+1}))})}$$

$$-\sum_{i=r+1}^{m+r}c(t_i)-2\sum_{i=r+1}^{m+r}(\frac{B(t_i)}{(1-B(t_i))}-\frac{B(t_i)h(t_i)}{(1-B(t_i))}-\frac{f(t_i)h(t_i)}{(1-B(t_i))^2})$$

$$+\sum_{i=r+1}^{m+r}(\frac{J(t_i)(1-B(t_i))R_iC(t_i)(-\frac{A(t_i)C(t_i)}{(1-B(t_i))}-\frac{a(t_i)C(t_i)}{(1-B(t_i))^2})}{\alpha}$$

$$+\frac{(1-\alpha)R_iC(t_i)(-\frac{A(t_i)C(t_i)}{(1-B(t_i))}-\frac{a(t_i)C(t_i)}{(1-B(t_i))^2})}{\alpha}$$

$$+\frac{J(t_i)R_i(1-B(t_i))(-\frac{A(t_i)c(t_i)}{(1-B(t_i))}-\frac{a(t_i)c(t_i)}{(1-B(t_i))^2}+\frac{A(t_i)h(t_i)}{(1-B(t_i))}+\frac{3a(t_i)h(t_i)}{(1-B(t_i))^2}+\frac{2d(t_i)h(t_i)}{(1-B(t_i))^3})}{\alpha})\qquad(1-22)$$

Where,

$$Q(t_{r+1})=e^{-2(\lambda t_{r+1})^\beta}$$
$$H(t_{r+1})=\alpha e^{-3(\lambda t_{r+1})^\beta}$$
$$J(t_{r+1})=e^{(\lambda t_{r+1})^\beta}$$
$$K(t_{r+1})=\alpha e^{-2(\lambda t_{r+1})^\beta}$$
$$L(t_{r+1})=e^{-2(\lambda t_{r+1})^\beta}$$
$$N(t_{r+1})=\beta t^2(\lambda t)^{\beta-2}$$

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$$S(t_{r+1}) = \alpha(1-\alpha)e^{-3(\lambda t_{r+1})^{\beta}}$$
$$a(t_{r+1}) = \alpha(1-\alpha)e^{-2(\lambda t_{r+1})^{\beta}}$$
$$b(t_{r+1}) = \beta t_{r+1}\log[\lambda t_{r+1}](\lambda t_{r+1})^{\beta-1}$$
$$c(t_{r+1}) = \log[\lambda t_{r+1}]^{2}(\lambda t_{r+1})^{\beta}$$
$$d(t_{r+1}) = \alpha(1-\alpha)^{2}e^{-3(\lambda t_{r+1})^{\beta}}$$
$$e(t_{r+1}) = \beta t_{r+1}\log[\lambda t_{r+1}](\lambda t_{r+1})^{2\beta-1}$$
$$f(t_{r+1}) = (1-\alpha)^{2}e^{-2(\lambda t_{r+1})^{\beta}}$$
$$g(t_{r+1}) = t(\lambda t_{r+1})^{\beta-1}$$
$$h(t_{r+1}) = \log[\lambda t_{r+1}](\lambda t_{r+1})^{2\beta}$$
$$w(t_{r+1}) = \beta^{2}t^{2}(\lambda t_{r+1})^{2\beta-2}$$

For $\alpha > 0$, the maximum likelihood estimators ($\hat{\alpha},\hat{\lambda},\hat{\beta}$) of (α,λ,β) ,are consistent estimators, and $\sqrt{n}\left(\hat{\alpha}-\alpha,\hat{\lambda}-\lambda,\hat{\beta}-\beta\right)$ is asymptotically to normal with mean vector 0 and variance-covariance matrix I^{-1}.

Asymptotic variance covariance m*atrix*

$$\hat{\text{var}}\left(\hat{\alpha},\hat{\lambda},\hat{\beta}\right)=\begin{pmatrix} V\left(\hat{\alpha}\right) & Cov\left(\hat{\alpha},\hat{\lambda}\right) & Cov\left(\hat{\alpha},\hat{\beta}\right) \\ Cov\left(\hat{\alpha},\hat{\lambda}\right) & V\left(\hat{\lambda}\right) & Cov\left(\hat{\lambda},\hat{\beta}\right) \\ Cov\left(\hat{\alpha},\hat{\beta}\right) & Cov\left(\hat{\lambda},\hat{\beta}\right) & V\left(\hat{\beta}\right) \end{pmatrix} \approx \begin{pmatrix} \left.\frac{\partial^2 l}{\partial\alpha^2}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\alpha\partial\beta}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} \\ \left.\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\lambda^2}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\lambda\partial\beta}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} \\ \left.\frac{\partial^2 l}{\partial\beta\partial\alpha}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\beta\partial\lambda}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} & \left.\frac{\partial^2 l}{\partial\beta^2}\right|_{\hat{\alpha},\hat{\lambda},\hat{\beta}} \end{pmatrix}.$$

Numerical experiments and data analysis

In this section, we present some results from generating progressive censored type II data from extended Weibull distribution by using *Mathematica* to observe how the MLEs perform for different sampling schemes and for different sample sizes, The asymptotic variance covariance matrix is computed also. We have taken n=50, 100 and 150, m=20, 40 and 60.

Different sampling schemes. Different values of the parameters are takenα=1.8, 2.0, and 3.0, λ=0.5, 1.5, and 3, β=0.5, 1.5 and3

In each case, we have calculated the MLEs. We replicate the process 1000 times and compute the average biases and standard deviations of the different estimates.

The scheme (*n*, *m*, *r*),

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Scheme[1] $n=50$, $m=20$, $r=0$, $R_1= R_2= R_3=$……………………………… $R_{19}=1$ $R_{20}=11$.

Scheme [2] $n=100$, $m=40$, $r=0$, $R_1= R_2= R_3=$……………………… $R_{39}=1$ $R_{40}=2$

Scheme [3] $n=150$, $m=60$, $r=0$, $R_1= R_2= R_3=$……………………………… $R_{59}=1$ $R_{60}=31$.

Scheme[4] $n=200$, $m=80$, $r=0$, $R_1= R_2= R_3=$ ……………………………… $R_{79}=1$, $R_{80}=41$.

Table 1: Estimation of parameters when $\alpha=3$, $\lambda=1.5$, $\beta=0.5$.

Scheme	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{var}(\hat{\alpha},\hat{\lambda},\hat{\beta})$
(50, 20, 0)	2.9731	1.7182	**0.**5899	$\begin{pmatrix} 0.2694 & 0.1383 & 0.02632 \\ 0.1383 & 0.05263 & 0.01195 \\ 0.02632 & 0.01195 & 0.00224 \end{pmatrix}$
(100,40,0)	2.8718	1.6092	2.7947	$\begin{pmatrix} 0.4225 & 0.0721 & 0.0418 \\ 0.0721 & 0.0165 & 0.00835 \\ 0.0418 & 0.00835 & 0.0061 \end{pmatrix}$

Table 2 Estimation of parameters using *Scheme[3]*

α	λ	β	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{var}(\hat{\alpha},\hat{\lambda},\hat{\beta})$
1.8	0.5	1.5	1.718	0.705	1.455	$\begin{pmatrix} 0.05829 & 0.02384 & 0.13491 \\ 0.02384 & 0.01214 & 0.06020 \\ 0.13491 & 0.06020 & 0.35797 \end{pmatrix}$
2	3	3	2.324	2.9	3.1	$\begin{pmatrix} 0.0040 & 0.02629 & 0.05547 \\ 0.02629 & 0.1575 & 0.3119 \\ 0.05547 & 0.3119 & 0.6614 \end{pmatrix}$

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Table 3 Estimation of parameters using *Scheme[4]*

α	λ	β	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{var}(\hat{\alpha},\hat{\lambda},\hat{\beta})$
1.8	0.5	1.5	2.2	0.5472	1.5	$\begin{pmatrix} 0.0745 & 0.1844 & 0.0593 \\ 0.1844 & 0.2432 & 0.1765 \\ 0.0593 & 0.1765 & 0.0573 \end{pmatrix}$
2	3	3	2.2	2.9174	3.1188	$\begin{pmatrix} 0.3120 & 0.0437 & 0.4765 \\ 0.0437 & 0.0075 & 0.0712 \\ 0.4765 & 0.0712 & 0.7111 \end{pmatrix}$

Bayesian Estimation

Here we consider the bays estimation of the unknown parameters when both are unknown. Here we assume gamma prior for α, λ and β

$$f_1(\alpha) \propto \alpha^{b-1} e^{-a\alpha}$$
$$f_2(\lambda) \propto \lambda^{d-1} e^{-c\lambda}$$
$$f_3(\beta) \propto \beta^{k-1} e^{-m\beta}$$

If t_1, t_2 , ……… t_n are the random sample of size n then the likelihood function is given by

$$L(t,\alpha,\lambda,\beta) = \frac{(\alpha\lambda\beta)^n \prod_{i=1}^{n} (\lambda t_i)^{\beta-1} e^{-\prod_{i=1}^{n} (\lambda t_i)^{\beta}}}{\prod_{i=1}^{n} (1-(1-\alpha) e^{-(\lambda t_i)^{\beta}})^2} \qquad (1-22)$$

The joint posterior density function of α,λ and β can be written as

$$\Pi(\alpha,\lambda,\beta) = \frac{L(t,\alpha,\lambda,\beta) f_1(\alpha) f_2(\lambda) f_3(\beta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(t,\alpha,\lambda,\beta) f_1(\alpha) f_2(\lambda) f_3(\beta) d(\alpha) d(\lambda) d(\beta)} \qquad (1-23)$$

Therefore, the bays estimator of any function of α, λ and β say g (α,λ,β)

$$E\,\alpha,\lambda,\beta|t\ (g(\alpha,\lambda,\beta)) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty L(t,\alpha,\lambda,\beta) f_1(\alpha) f_2(\lambda) f_3(\beta) d(\alpha) d(\lambda) d(\beta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(t,\alpha,\lambda,\beta) f_1(\alpha) f_2(\lambda) f_3(\beta) d(\alpha) d(\lambda) d(\beta)} \qquad (1-24)$$

It is not possible to compute (1-24) analytically in this case therefore we use MCMC methods to find the bays estimator of α, λ and β using α=3, λ=1and β=2

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

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	mean	SD	mc error	2.5%	median	97.5%
$\hat{\alpha}$	2.742	1.376	0.2156	0.805	2.518	6.002
$\hat{\beta}$	2.202	0.3523	0.0533	1.622	2.177	2.94
$\hat{\lambda}$	1.004	0.1593	0.02584	0.725	0.9945	1.348

CONCLUSION

We can say while the Newton-Raphson method did not converge in many cases or converged to strange unacceptable results, the bisection method, while slow, gave good results under different progressive sampling schemes. As a rule of thumb, better results are obtained when the total number of observations is at least four times as much as the number of parameters. Also, different sampling schemes gave comparable results within the sample sizes, *n*, considered.

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