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ESTIMATION OF PARAMETERS OF THE MARSHALL-OLKIN EXTENDED LOG-LOGISTIC DISTRIBUTION FROM PROGRESSIVELY CENSORED SAMPLES

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ABSTRACT: *In this paper, a new class of Log-Logistic distribution using Marshall-Olkin transformation is introduced. Its characterization and statistical properties are obtained. And The estimation of parameters of distribution under progressive censoring is investigated, maximum likelihood estimators of the unknown parameters are obtained using statistical software (Mathematica), MLE performs for different sampling schemes with different sample sizes is observed, and the asymptotic variance covariance matrix is computed also.*

KEYWORDS: Extended Log-Logistic distribution · progressive type II censoring · Maximum likelihood estimators · Asymptotic variance covariance matrix.

INTRODUCTION

The Log-Logistic distribution (known as the Fisk distribution in economics) is the probability distribution of a random variable whose logarithm has a logistic distribution .It has attracted a wide applicability in survival and reliability over the last few decades, particularly for events whose rate increases initially and decrease later, for example mortality from cancer following diagnosis or treatment, see [10]. It has also been used in hydrology to model stream flow and precipitation, see [14] and [6], for modeling good frequency, see [1]. [4] Used this model to describe the thermal inactivation of Clostridium botulinum 213B at temperatures below 121:1 _C. Furthermore, it is applied in economics as a simple model of the distribution of wealth or income, see [8].The cumulative distribution function F(x) and density function f(x) of Log-Logistic distribution are given by

$$F(x) = \frac{x^{\beta}}{\alpha^{\beta} + x^{\beta}} \quad , \qquad x > o \tag{1-1}$$

And

$$f(x) = \frac{\alpha^{\beta}\, \beta\, x^{\beta-1}}{(x^{\beta} + \alpha^{\beta})^{2}} \ , \ \ x > \tag{1-2}$$

Where α > 0 is the scale parameter and is also the median of the distribution, β > 0 is the shape parameter .When β > 1, the Log-Logistic distribution is unimodal. It is similar in shape to the log-normal distribution but has heavier tails. Its cumulative distribution function can be written in closed form, unlike that of the log-normal. [7] Used the ratio of maximized likelihood to consider the discrimination procedure between the two distribution functions.

On the other hand, [12] introduced a new family of survival functions which is obtained by adding a new parameter γ > 0 to an existing distribution. The new parameter will result in flexibility in the distribution. Let $\bar{F}(x) = 1 - F(x)$ be the survival function of a random variable X.

Then

$$\bar{G}(x) = \frac{\gamma F(x)}{1-(1-\gamma)\bar{F}(x)} \tag{1-3}$$

Is a proper survival function. $\bar{G}(x)$is called Marshall-Olkin family of distributions. If γ= 1, we have that G = F. The density function corresponding to (3) is given by

$$g(x) = \frac{\gamma f(x)}{(1-(1-\gamma)\bar{F}(x))^2} \tag{1-4}$$

And the hazard rate function is given by

$$h(x) = \frac{h_F(x)}{1-(1-\gamma)\bar{F}(x)} \tag{1-5}$$

Where $h_F(x)$ is the hazard rate function of the original model with distribution F.By using the Marshall-Olkin transformation (1-3), several researchers have considered various distribution extensions in the last few years. [12] generalized the exponential and Weibull distributions using this technique. [2] introduced Marshall-Olkin extended semi Pareto model for Pareto type III and established its geometric extreme stability. Semi-Weibull distribution and generalized Weibull distributions are studied by [3]. Ghit any et al. (2005) conducted a detailed study of Marshall-Olkin Weibull distribution, that can be obtained as a compound distribution mixing with exponential distribution, and apply it to model censored data. Marshall-Olkin Extended Lomax Distribution was introduced by [9]. [11] Investigated Marshall-Olkin q-Weibull distribution and its max-min processes .In this paper, we use the Marshall-Olkin transformation to define a new model, so-called the Marshall-Olkin Log-Logistic distribution, which generalizes the Log-Logistic model. We aim to reveal some statistical properties of the proposed model and estimation of its parameters.

Density function

Let t follows Log-Logistic distribution, then its survival function is given by

$$\bar{F}(t) = 1 - \frac{t^{\beta}}{\alpha^{\beta} + t^{\beta}} \quad , \qquad \frac{\alpha^{\beta}}{\alpha^{\beta} + t^{\beta}}$$

Substituting it in (1-3) we obtain a survival function of Marshall-Olkin Log-Logistic distribution denoted by

$$\bar{G}(t) = \frac{\alpha^{\beta}\,\gamma}{t^{\beta} + \alpha^{\beta}\,\gamma} \qquad \alpha, \beta, \gamma, t \; > 0 \tag{1-6}$$

The corresponding density function is given by

$$g(t) = \frac{\alpha^{\beta}\,\beta\gamma t^{\beta-1}}{(t^{\beta} + \alpha^{\beta}\,\gamma)^2} \qquad \alpha, \beta, \gamma, t \; > \tag{1-7}$$

If $\gamma = 1$, we obtain the Log-Logistic distribution with parameter α, β>0.This distribution contains the Log-Logistic distribution as a particular case .The following theorem gives some conditions under which the density function(1-7) is decreasing or unimodal.

Maximum Likelihood Estimation

In case complete data

Consider a random sample consisting of n observations from the log logistic distribution and the likelihood function of this sample is

$$L\left(t_{1,}t_2, \cdots, t_n; \beta, \alpha, \gamma\right) = \prod_{i=1}^{n} f(t_i) \tag{1-8}$$

$$L(\boldsymbol{t}, \beta, \alpha, \gamma) = \frac{\prod_{i=1}^{n}(\beta\gamma\alpha^{\beta} {t_i}^{\beta-1})}{\prod_{i=1}^{n}\left(t_i^{\beta} + \alpha^{\beta}\gamma\right)^2} \tag{1-9}$$

On taking logarithms of (1-9) we get,

$$LL = n(\log(\beta\gamma)) + n\left(\log[\alpha^{\beta}]\right) + (\beta - 1)\sum_{i=1}^{n}\log[t_i] - 2\sum_{i=1}^{n}\log[t_i^{\beta} + \left(\alpha^{\beta}\gamma\right)] \tag{1-10}$$

Differentiating (1-10) with respect to β, αand γ and equating to zero are:

$$\frac{\partial LL}{\partial \beta} = \frac{n}{\beta} + nlog[\alpha] + \sum_{i=1}^{n}\log[t_i] - 2\sum_{i=1}^{n}\frac{\alpha^{\beta}\gamma\log[\alpha] + \log[t_i]\, t_i^{\beta}}{\alpha^{\beta}\gamma t_i^{\beta}} \tag{1-11}$$

$$\frac{\partial LL}{\partial \alpha} = \frac{n\beta}{\alpha} - 2\sum_{i=1}^{n}\frac{\alpha^{\beta-1}\beta\gamma}{\alpha^{\beta}\gamma t_i^{\beta}} \tag{1.12}$$

$$\frac{\partial LL}{\partial \gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n}\frac{\alpha^{\beta}}{\alpha^{\beta}\gamma t_i^{\beta}} \tag{1-13}$$

The maximum likelihood estimate (MLE) ($\hat{\beta},\hat{\alpha},\hat{\gamma}$) of (β,α,γ) is obtained by solving the nonlinear likelihood equations (1-11), (1-12) and (1-13). These equations cannot be solved analytically and popular software can be used to solve these equations numerically.

In case progressive type two right censored data.

Let $t_1, t_2, \dots\dots\dots t_n$be independent and identically distributed random lifetimes of n items. A type–ll progressively right censored sample may be obtained in the following way: the failure time of the first r_1 items to fail are not observed; at the time of the (r+1)st failure, denoted with t_{r+1}, R_{r+1} of the remaining units are withdrawn from the test randomly. At the time of next failure, denoted with t_2, R_{r+2} surviving items are removed at random from the remaining items, and so on. At the time of the mth failure, where m is a predetermined number, all the remaining R_{r+m} items are censored.

Therefore, a progressively type-ll right censoring scheme is specified by a sequence integers n, m and $r_1,\ldots\ldots,r_{m-1}$ with the constraints

$n- m- r_1 -\ldots r_{m-1}\geq 0,\ n\geq m\geq 1.$

The likelihood function to be maximized when general progressive type- II censored sample based on n independent from *F (t)* is

$$L(\theta)=c^{*}\left[F\left(t_{r+1},\theta\right)\right]^{r}\prod_{i=r+1}^{m+r}f\ \left(t_{i},\theta\right)\left[1-F\left(t_{i},\theta\right)\right]^{R_{i}} \tag{1-14}$$

Where,

$$,t_{r+1}<t_{r+2}<\ldots<t_{m},$$

$$c^{*}=\frac{n!}{r!\left(n-r-1\right)!}\left(n-R_{r+1}-r-1\right)\ldots\left(n-R_{r+1}-\ldots-R_{m-1}-m+1\right),$$

The likelihood function given in Eq. (1.14) can be expressed as:

$$L\left(t;\beta,\alpha,\gamma\right)=c^{*}\left(\frac{t_{r+1}}{t_{r+1}+\alpha^{\beta}\gamma}\right)^{r}\prod_{i=1}^{m+r}\frac{\left(\beta\gamma\right)\alpha^{\beta}(t_{i})^{\beta-1}}{\left(t_{i}+(\alpha^{\beta}\gamma)\right)^{2}} \times\left(\frac{(t_{i})^{\beta}}{(t_{i})^{\beta}+\alpha^{\beta}\gamma}\right)^{R_{i}} \tag{1-15}$$

$$L\left(t;\beta,\alpha,\gamma\right)=c^{*}(\beta\gamma)^{m+r}\left(\frac{t_{r+1}}{t_{r+1}+\alpha^{\beta}\gamma}\right)^{r}\prod_{i=1}^{m+r}\frac{\alpha^{\beta}(t_{i})^{\beta-1}}{\left(t_{i}+(\alpha^{\beta}\gamma)\right)^{2}} \times\left(\frac{(t_{i})^{\beta}}{(t_{i})^{\beta}+\alpha^{\beta}\gamma}\right)^{R_{i}} \tag{1-16}$$

The log-likelihood function, denoted by $l\left(t,\beta,\alpha,\gamma\right)$, takes the form

$$l\left(t;\beta,\alpha,\gamma\right)=\text{constant}+\left(m+r\right)\log\left[\beta\gamma\right]+r\log\left[\frac{t_{r+1}}{t_{r+1}+\alpha^{\beta}\gamma}\right]+$$
$$+(m+r)\log[\alpha^{\beta}]+(\beta-1)\sum_{i=1}^{m+r}\log[t_i]-2\sum_{i=1}^{m+r}\log\left[t_i^{\ \beta}+\alpha^{\beta}\gamma\right]$$
$$+\sum_{i=1}^{m+r}R_i\log\left[\frac{\alpha^{\beta}\gamma}{t_i^{\ \beta}+\alpha^{\beta}\gamma}\right] \qquad (1\text{-}17)$$

The three equations obtained by differentiating (1.17) and equating to zero are

$$\frac{\partial\ell}{\partial\beta}=\frac{m+r}{\beta}+(m+r)\log[\alpha]-\frac{r\alpha^{\beta}\gamma\log[\alpha]}{\alpha^{\beta}\gamma+t_{r+1}}+$$
$$\sum_{i=1}^{m+r}\log[t_i]-2\sum_{i=1}^{m+r}\frac{\alpha^{\beta}\gamma\log[\alpha]+\log[t_i]\ t_i^{\ \beta}}{\alpha^{\beta}\gamma+t_i^{\ \beta}} \qquad (1\text{-}18)$$
$$\sum_{i=1}^{m+r}\frac{\alpha^{-\beta}R_i(\alpha^{\beta}\gamma+t_i)(\frac{\alpha^{\beta}\gamma\log[\alpha]}{(\alpha^{\beta}\gamma+t_i)}-\frac{\alpha^{\beta}\gamma(\alpha^{\beta}\gamma\log[\alpha]+\log[t_i]\ t_i^{\ \beta}}{(\alpha^{\beta}\gamma+t_i)^2}}{\gamma}=0,$$

$$\frac{\partial\ell}{\partial\alpha}=\frac{m+r}{\alpha}+\frac{(m+r)\beta}{\alpha}-\frac{r\alpha^{\beta-1}\beta\,\gamma}{\alpha^{\beta}\gamma+t_{r+1}}-2\sum_{i=1}^{m+r}\frac{\alpha^{\beta-1}\beta\,\gamma}{\alpha^{\beta}\gamma+t_i}$$
$$\sum_{i=1}^{m+r}\frac{\alpha^{-\beta}R_i(\alpha^{\beta}\gamma+t_i^{\ \beta})(-\frac{\alpha^{2\beta-1}\beta\,\gamma^2}{(\alpha^{\beta}\gamma+t_i^{\ \beta})^2}+\frac{\alpha^{\beta-1}\beta\,\gamma}{(\alpha^{\beta}\gamma+t_i^{\ \beta})})}{\gamma}=0, \qquad (1\text{-}19)$$

$$\frac{\partial\ell}{\partial\gamma}=-\frac{r\alpha^{\beta}}{\alpha^{\beta}+t_{r+1}}-2\sum_{i=1}^{m+r}\frac{\alpha^{\beta}}{\alpha^{\beta}\gamma+t_i^{\beta}}+\sum_{i=1}^{m+r}\frac{\alpha^{-\beta}R_i(\alpha^{\beta}\gamma+t_i^{\beta})(-\frac{\alpha^{2\beta}\gamma}{(\alpha^{\beta}\gamma+t_i^{\beta})^2}+\frac{\alpha^{\beta}}{(\alpha^{\beta}\gamma+t_i^{\beta})}}{\gamma}=0, \qquad (1\text{-}20)$$

The maximum likelihood estimate (MLE) $\left(\hat{\beta},\hat{\alpha},\hat{\gamma}\right)$ of $\left(\beta,\alpha,\gamma\right)$ is obtained by solving the nonlinear likelihood equations (1-18), (1-19) and (1-20). These equations cannot be solved analytically and popular software can be used to solve the equations numerically.

4. Fisher information matrix $I(\theta)$

$$I(\theta) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\gamma}\right) \\ E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right) \\ E\left(\frac{\partial^2 l}{\partial\gamma\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\gamma\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\gamma^2}\right) \end{pmatrix}.$$

The elements of the sample information matrix, for progressively type II censored will be

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta^2} &= -\frac{m+r}{\beta^2} + \frac{rC}{F(t_{r+1})} - 2\sum_{i=1}^{m+r}\left(-\frac{(A+D(t_i))^2}{(G(t_i))^2} + \frac{B+Q(t_i)}{G(t_i)}\right) \\ &\sum_{i=1}^{m+r}\Bigg(-\frac{H\,R_i(G(t_i))\left(\frac{A}{G(t_i)} - \frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2}\right)}{\gamma} - \frac{\alpha^{-\beta}R_i(A+D(t_i))\left(\frac{A}{G(t_i)} - \frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2}\right)}{\gamma} \\ &+ \frac{\alpha^{-\beta}R_i(G(t_i))\left(\frac{B}{G(t_i)} - \frac{2A(A+D(t_i))}{G(t_i))^2} + \frac{2\alpha^{\beta}\gamma(A+D(t_i))^2}{G(t_i))^3} - \frac{\alpha^{\beta}\gamma(B+Q(t_i))}{G(t_i))^2}\right)}{\gamma}\Bigg)\end{aligned} \qquad (1\text{-}21)$$

$$\frac{\partial^2 \ell}{\partial\beta\,\partial\alpha}=\frac{m+r}{\alpha}+\frac{rN}{(F(t_{r+1}))^2}-\frac{r\,\alpha^{\beta-1}\gamma}{F(t_{r+1})}-\frac{rq}{F(t_{r+1})}-2\sum_{i=1}^{m+r}(\frac{\alpha^{\beta-1}\gamma+q}{F(t_i)}-\frac{\alpha^{\beta-1}\gamma\,\beta(A+D(t_i))}{F(t_i)^2})$$
$$+\sum_{i=1}^{m+r}(\frac{\beta R_i(\frac{A}{G(t_i)}-\frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2})}{\alpha}-\frac{\alpha^{\beta-1}\beta R_i G(t_i)(\frac{A}{G(t_i)}-\frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2})}{\gamma}$$
$$+\frac{\alpha^{-\beta}R_i G(t_i)(-\frac{N}{(G(t_i))^2}-\frac{\alpha^{\beta}\gamma(\gamma\,\alpha^{\beta-1}+q)}{(G(t_i))^2}+\frac{2\alpha^{\beta}\gamma(A+D(t_i))^2}{(G(t_i))^3}-\frac{\alpha^{\beta}\gamma(B+Q(t_i))}{(G(t_i))^2}}{\gamma}) \tag{1-22}$$

$$\frac{\partial^2 \ell}{\partial\beta\,\partial\gamma}=\frac{rc}{(F(t_{r+1}))^2}-\frac{ra}{F(t_{r+1})}-2\sum_{i=1}^{m+r}(\frac{a}{G(t_i)}-\frac{\alpha^{\beta}(A+D(t_i))}{(G(t_i))^2})$$
$$+\sum_{i=1}^{m+r}(\frac{\alpha^{-\beta}R_i(G(t_i))(-\frac{2c}{(G(t_i))^2}+\frac{a}{G(t_i)}+\frac{2\alpha^{2\beta}\gamma(A+D(t_i))}{(G(t_i))^3}-\frac{\alpha^{\beta}(A+D(t_i))}{(G(t_i))^2})}{\gamma}$$
$$\frac{+R_i(\frac{A}{G(t_i)}-\frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2})}{\gamma}-\frac{\alpha^{-\beta}R_i(G(t_i))(\frac{A}{G(t_i)}-\frac{\alpha^{\beta}\gamma(A+D(t_i))}{(G(t_i))^2})}{\gamma^2}) \tag{1-23}$$

$$\frac{\partial^2 \ell}{\partial\alpha^2}=-\frac{m+r}{\alpha^2}-\frac{(m+r)\beta}{\alpha^2}+\frac{rb}{(F(t_{r+1}))^2}-\frac{rd}{F(t_{r+1})}-2\sum_{i=1}^{m+r}(-\frac{b}{(G(t_i))^2}+\frac{d}{G(t_i)}$$
$$+\sum_{i=1}^{m+r}(\frac{\beta R_i(-\frac{f}{(G(t_i))^2}+\frac{g}{G(t_i)})}{\alpha}-\frac{\beta\,\alpha^{-\beta-1}R_i(G(t_i))(-\frac{f}{(G(t_i))^2}+\frac{g}{G(t_i)})}{\gamma}$$
$$+\frac{\alpha^{-\beta}R_i(G(t_i))(\frac{2h}{(G(t_i))^3}-\frac{b}{(G(t_i))^2}-\frac{j}{(G(t_i))^2}+\frac{d}{(G(t_i))})}{\gamma}) \tag{1-24}$$

$$\frac{\partial^2 \ell}{\partial\alpha\,\partial\gamma}=\frac{rk}{(F(t_{r+1}))^2}-\frac{r\alpha^{\beta-1}\beta}{F(t_{r+1})}-2\sum_{i=1}^{m+r}(\frac{\alpha^{-\beta}R_i(G(t_i))(\frac{2J}{(G(t_i))^3}-\frac{3k}{(G(t_i))^2}+\frac{\beta\alpha^{\beta-1}}{G(t_i)})}{\gamma}$$

$$+\sum_{i=1}^{m+r}\left(\frac{\alpha^{-\beta}R_i(G(t_i))\left(\frac{2J}{(G(t_i))^3}-\frac{3k}{(G(t_i))^2}+\frac{\alpha^{\beta-1}\beta}{G(t_i)}\right)}{\gamma}+\frac{R_i\left(-\frac{f}{(G(t_i))^2}+\frac{g}{G(t_i)}\right)}{\gamma}-\frac{\alpha^{-\beta}R_i(G(t_i))\left(-\frac{f}{(G(t_i))^2}+\frac{g}{G(t_i)}\right)}{\gamma^2}\right) \quad (1\text{-}25)$$

$$\frac{\partial^2\ell}{\partial\gamma^2}=\frac{r\,\alpha^{2\beta}}{(F(t_{r+1}))^2}-2\sum_{i=1}^{m+r}-\frac{\alpha^{2\beta}}{(G(t_i))^2}+\sum_{i=1}^{m+r}\left(\frac{\alpha^{-\beta}R_i(G(t_i))\left(\frac{2\alpha^{3\beta}\gamma}{(G(t_i))^3}-\frac{2\alpha^{2\beta}}{(G(t_i))^2}\right)}{\gamma}+\frac{R_i\left(-\frac{\alpha^{2\beta}\gamma}{(G(t_i))^2}+\frac{\alpha^{\beta}}{G(t_i)}\right)}{\gamma}-\frac{\alpha^{-\beta}R_i(G(t_i))\left(-\frac{\alpha^{2\beta}\gamma}{(G(t_i))^2}+\frac{\alpha^{\beta}}{G(t_i)}\right)}{\gamma^2}\right) \quad (1\text{-}26)$$

Where,

$A=\alpha^{\beta}\gamma\log[\alpha]$	$B=\alpha^{\beta}\gamma\log[\alpha]^2$
$C=\alpha^{2\beta}\gamma^2\log[\alpha]^2$	$D(t_{r+1})=\log[t_{r+1}]t_{r+1}^{\beta}$
$Q(t_{r+1})=\log[t_{r+1}]^2\,t_{r+1}^{\beta}$	$F(t_{r+1})=\alpha^{\beta}\gamma+t_{r+1}$
$G(t_{r+1})=\alpha^{\beta}\gamma+t_{r+1}^{\beta}$	$H=\alpha^{-\beta}\log[\alpha]$
$J=\alpha^{3\beta-1}\beta\gamma^2$	$N=\alpha^{2\beta-1}\beta\gamma^2\log[\alpha]$
$a=\alpha^{\beta}\log[\alpha]$	$b=\alpha^{2\beta-2}\beta^2\gamma^2$
$c=\alpha^{2\beta}\gamma\log[\alpha]$	$d=(\beta-1)\alpha^{\beta-2}\beta\gamma$
$f=\alpha^{2\beta-1}\beta\gamma^2$	$g=\alpha^{\beta-1}\beta\gamma$
$h=\alpha^{2\beta-2}\beta^2\gamma^3$	$j=(2\beta-1)\alpha^{2\beta-2}\beta\gamma^2$
$k=\alpha^{2\beta-1}\beta\gamma$	$q=\alpha^{\beta-1}\beta\gamma\log[\alpha]$

For $\gamma>0$, the maximum likelihood estimators $(\hat{\beta},\hat{\alpha},\hat{\gamma})$ of (β,α,γ) are consistent estimators, and $\sqrt{n}\left(\hat{\beta}-\beta,\hat{\alpha}-\alpha,\hat{\gamma}-\gamma\right)$ is asymptotically to normal with mean vector 0 and variance-covariance matrix I^{-1}.

Asymptotic variance covariance m*atrix*

$$\hat{var}\left(\hat{\beta},\hat{\alpha},\hat{\gamma}\right)=\begin{pmatrix} V\left(\hat{\beta}\right) & Cov\left(\hat{\beta},\hat{\alpha}\right) & Cov\left(\hat{\beta},\hat{\gamma}\right) \\ Cov\left(\hat{\alpha},\hat{\beta}\right) & V\left(\hat{\alpha}\right) & Cov\left(\hat{\alpha},\hat{\gamma}\right) \\ Cov\left(\hat{\gamma},\hat{\beta}\right) & Cov\left(\hat{\gamma},\hat{\alpha}\right) & V\left(\hat{\gamma}\right) \end{pmatrix} \approx \begin{pmatrix} \left.\frac{\partial^2 l}{\partial\beta^2}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\beta\partial\alpha}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\beta\partial\gamma}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} \\ \left.\frac{\partial^2 l}{\partial\alpha\partial\beta}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\alpha^2}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} \\ \left.\frac{\partial^2 l}{\partial\gamma\partial\beta}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\gamma\partial\alpha}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} & \left.\frac{\partial^2 l}{\partial\gamma^2}\right|_{\hat{\beta},\hat{\alpha},\hat{\gamma}} \end{pmatrix}.$$

A Numerical example and data analysis

In this section, we present some results from generating progressive censored type II data from extended logistic distribution by using *Mathematica* to observe how the MLEs perform for different sampling schemes and for different sample sizes, The asymptotic variance covariance matrix is computed also. We have taken n=50, 100, 150 and200, m=20, 30,50and70

Different sampling schemes. Different values of the parameters are taken$\beta=3$, $\alpha=4$ and $\gamma=2$, $\beta=1.5$, $\alpha=3$ and $\gamma=1$in each case, we have calculated the MLEs. We replicate the process 1000 times and compute the average biases and standard deviations of the different estimates.

The scheme (n, m, r),

Scheme [1] n=50, m=20, r=0, $R_1= R_2=1, R_3=2, R_4=1, R_5=3, R_6=2, R_7=R_8=1, R_9=3, R_{10}= R_{11}=1, R_{12}=3, R_{13}=2, R_{14}=0, R_{15}= R_{16}= R_{17}=1, R_{18}= R_{19}=2, R_{20}=1$

Scheme[2]n=100, m=30, r=0, $R_1= R_2=3, R_3=6, R_4= R_5=1, R_6=4, R_7=R_8=R_9=R_{10}=3 R_{11}=1, R_{12}=2, R_{13}=1, R_{14}=R_{15}=R_{16}=R_{17}=2,$

$R_{18}=6, R_{19}=5, R_{20}= R_{21}=R_{22}=0, R_{23}=1, R_{24}=2, R_{25}=3, R_{26}=4, R_{27}= R_{28}=1, R_{29}=4, R_{30}=1.$

scheme[3] $n=150$, $m=50$, $r=0$, $R_1=0$, $R_2=1$, $R_3=0$, $R_4= R_5= R_6=2$, $R_7=4$, $R_8=2$,$R_9=3$,$R_{10}=1$, $R_{11}=3$,$R_{12}=5$,$R_{13}=4$, $R_{14}=5$, $R_{15}=0$,$R_{16}=8$,$R_{17}=4$,$R_{18}=0$,$R_{19}=1$,$R_{20}=R_{21}=2$,$R_{22}=0$,$R_{23}=3$,$R_{24}=2$,

$R_{25}=4$,$R_{26}=R_{27}=1$,$R_{28}=2$,$R_{29}=0$,$R_{30}=1$,$R_{31}=2$,$R_{32}=4$,$R_{33}=1$,$R_{34}=3$,

$R_{35}=0$,$R_{36}=1$,$R_{37}=R_{38}=0$,$R_{39}=2$,$R_{40}=3$,$R_{41}=2$,$R_{42}=6$,$R_{43}=0$,$R_{44}=1$,

$R_{45}=2$, $R_{46}=0$, $R_{47}=R_{48}=1$, $R_{49}=4$, $R_{50}=2$.

Scheme[4] $n=200$, $m=70$, $r=0$, $R_1=0$, $R_2=4$, $R_3=2$, $R_4=0$,$R_5=3$,$R_6=R_7=2$,$R_8=R_9=1$,$R_{10}=4$,$R_{11}=R_{12}=2$,$R_{13}=4$,$R_{14}=R_{15}=1$

$R_{16}=R_{17}=0$,$R_{18}=2$,$R_{19}=R_{20}=1$,$R_{21}=2$,$R_{22}=0$,$R_{23}=R_{24}=2$,$R_{25}=1$,

$R_{26}=0R_{27}=R_{28}=2$,$R_{29}=4$, $R_{30}=R_{31}=1$, $R_{32}=5$,$R_{33}=R_{34}=1$, $R_{35}=2$, $R_{36}=3$,$R_{37}=0$,$R_{38}=2$,$R_{39}=3$,$R_{40}=2$,$R_{41}=1$,$R_{42}=3$,$R_{43}=4$,$R_{44}=1$,

$R_{45}=2$,$R_{46}=3$,$R_{47}=2$,$R_{48}=0$,$R_{51}=R_{52}=1$,$R_{53}=R_{54}=0$,$R_{55}=2$,$R_{56}=0$,

$R_{57}=R_{58}=1$, $R_{59}=4$, $R_{60}=R_{61}=1$, $R_{62}=2$, $R_{63}=R_{64}=0$,$R_{65}=3$,$R_{66}= R_{67}=2$, $R_{68}=6$, $R_{69}=5$, $R_{70}=2$.

Table 1: Estimation of parameters when $\beta=3$, $\alpha=4$, $\gamma=2$.

Scheme	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{var}\left(\hat{\beta},\hat{\alpha},\hat{\gamma}\right)$	
(50, 20, 0)	3.2499	4.0441	1.938	$\begin{pmatrix} 0.410822 & 0.052003 & 0.19120 \\ 0.052003 & 0.00703 & 0.02524 \\ 0.19120 & 0.02524 & 0.09686 \end{pmatrix}$	
(100,30,0)	3.1122	3.9698	2.210	$\begin{pmatrix} 0.60380 & 0.06399 & 0.47191 \\ 0.06399 & 0.007431 & 0.05230 \\ 0.47191 & 0.05230 & 0.39767 \end{pmatrix}$	

Table 2: Estimation of parameters using *Scheme[3]*

B	A	γ	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{var}\left(\hat{\beta},\hat{\alpha},\hat{\gamma}\right)$
3	4	2	3.0518	3.952	2.214	$\begin{pmatrix} 0.00363 & 0.02804 & 0.01097 \\ 0.02804 & 0.225423 & 0.08835 \\ 0.01097 & 0.08835 & 0.03669 \end{pmatrix}$
1.5	3	1	1.556	2.846	1.118	$\begin{pmatrix} 0.00378 & 0.02917 & 0.01031 \\ 0.02917 & 0.23270 & 0.08182 \\ 0.01031 & 0.08182 & 0.03054 \end{pmatrix}$

Table 3: Estimation of parameters using *Scheme[4]*

B	A	γ	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{var}\left(\hat{\beta},\hat{\alpha},\hat{\gamma}\right)$
3	4	2	3.249	4.0441	1.938	$\begin{pmatrix} 0.00913 & 0.00908 & 0.00804 \\ 0.00908 & 0.00934 & 0.00786 \\ 0.00804 & 0.00786 & 0.00746 \end{pmatrix}$
1.5	3	1	1.5310	2.8614	1.125	$\begin{pmatrix} 0.00378 & 0.02917 & 0.01031 \\ 0.02917 & 0.23270 & 0.08182 \\ 0.01031 & 0.08182 & 0.03054 \end{pmatrix}$

REFERENCES

[1] MI Ahmad, CD Sinclair, and A. Werritty, Log-logistic ood frequency Analysis, Journal of Hydrology 98 (1988), no. 3, 205{224.
[2] T. Alice and KK Jose, Marshall-olkinpareto processes, Far East Journal of Theoretical Statistics 9 (2003), no. 2, 117{132.
[3] , Marshall{olkin semi-weibull minication processes, Recent Adv Stat Theory Appl I (2005), 6{17.
[4] WA Anderson, PJ McClure, AC Baird-Parker, and MB Cole, theap- plication of a log-logistic model to describe the thermal inactivation of Clostridiumbotulinum 213b at temperatures below 121.1 c, Journal of Applied Microbiology 80 (1996), no. 3, 283{290.

[5] B.C. Arnold, N. Balakrishnan, and H.N. Nagaraja, A _rst course in order statistics, vol. 54, Society for Industrial Mathematics, 2008.

[6] F. Ashkar and S. Mahdi, Fitting the log-logistic distribution by generalized moments, Journal of Hydrology 328 (2006), no. 3, 694{703.

[7] A.K. Dey and D. Kundu, Discriminating between the log-normal and log- logistic distributions, Communications in Statistics-Theory and Methods 39 (2009), no. 2, 280{292.

[8] P.R. Fisk, The graduation of income distributions, Econometrical: Journal of The Econometric Society (1961), 171{185.

[9] ME Ghitany, FA Al-Awadhi, and LA Alkhalfan, Marshall{olkin extended lomax distribution and its application to censored data, Communications in Statistics Theory and Methods 36 (2007), no. 10, 1855{1866.

[10] R.C. Gupta, O. Akman, and S. Lvin, A study of log-logistic model in survival analysis, Biometrical Journal 41 (1999), no. 4, 431{443.

[11] KK Jose, S.R. Naik, and M.M. Risti_c, Marshall{olkin q-weibull distribution and max{min processes, Statistical papers 51 (2010), no. 4, 837{851.

[12] A.W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and weibull families, Biometrika 84 (1997), no. 3, 641{652.

[13] C.G. Ramesh and S. Kirmani, On order relations between reliability measures, Stochastic Models 3 (1987), no. 1, 149{156.

[14] MM Shoukri, IUH Mian, and DS Tracy, Sampling properties of estimators of the log-logistic distribution with application to canadian precipitation
data, Canadian Journal of Statistics 16 (1988), no. 3, 223{236.