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DOUBLE PERTURBATION COLLOCATION METHOD FOR SOLVING FRACTIONAL RICCATI DIFFERENTIAL EQUATIONS.

[1]**Taiwo O.A. and** [2]**Fesojaye, M. O.**

[1]*Department of Mathematics, Faculty of Physical Sciences,*
University of Ilorin, Ilorin, Nigeria

[D]*epartment of Mathematics and Statistics, School of Applied Arts and Science,*
Federal Polytechnic, Bida. Niger State, Nigeria.

ABSTRACT: *In this work, we proposed a computational technique called the Double Perturbation Collocation Method (DPCM) for the numerical solution of fractional Riccati differential equation. The DPCM requires the addition of a perturbation term to the approximate solution in terms of the shifted Chebyshev polynomials basis function. This function is substituted into a slightly perturbed fractional Riccati equation. The fractional derivative is in the Caputo sense. The resulting equation is simplified and then collocated at some equally spaced points. Thus resulted into system of equations which are then solved by implementing Gaussian elimination method for linear to obtain the unknown constants and for the case of nonlinear, Newton linearization scheme of appropriate orders are used to linearize. The values of the constants obtained are then substituted back into the perturbed approximate solution. Results obtained with DPCM compared favourably well with existing results in literature and the exact solutions where such existed in closed form. Some numerical examples are included to illustrate the accuracy, simplicity and computational cost of the method.*

KEYWORDS: collocation, double - perturbation, linearization, fractional, Riccati.

INTRODUCTION

Recently, considerable attention with keen interest has been drawn to the study of fractional differential equations. This is mainly due to the fact that physical phenomena are successfully modeled using fractional order differential equations. Hashimet. al (2009) stated that fractional differential equations have found applications in many problems in physics, mathematics and engineering.These areas include signal processing, control engineering, biosciences, fluid mechanics, electro-chemistry, diffusion processes, viscoelastic materials and so on. Podlubny (1999) observed that many researchers are attracted to this field of study because of its wide application areas in mathematics, economics and engineering. He (1999) stated that the fluid-dynamic traffic is modeled using fractional derivatives. For more details on fractional differential and integral equations see (Podlubny, 1999).

The Riccati differential equation was named after an Italian Nobleman called Count Jacopo Francesco Riccati (1676-1754) (Sweilam et. al 2014). Riccati equations are applicable in the random processing, optimal control and diffusion problems, stochastic realization theory, robust stabilization, network synthesis, financial mathematics. Reid (1972) gives detailed fundamental theory and applications of Riccati differential equations in the book entitled Riccati differential equations.Fractional Riccati differential equations arise in mathematical

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[1] Corresponding Author

modeling of many physical and engineering phenomena such as heat conduction, fluid flow, optical control, financial mathematics, acoustics, electro-magnetics, hydrology and biology processes (Bhrawy,(2014), Khader et. al (2014)). Consequently, getting accurate and efficient methods for solving problems of fractional derivatives has become an active research undertaking (Abdulaziz et al., (2008)).A good number of scientists has proposed different numerical methods for solving fractional derivatives.

He (1999, 2000) proposed the Variational Iterative Method for the solutions of linear and nonlinear problems of fractional order. Momani and Aslam (2006) employed the popular Adomian Decomposition Method to derive the analytic approximate solutions of the linear and nonlinear boundary value problems for fourth order fractional integral equations. Taiwo and Odetunde (2009) used iterative decomposition method to find the numerical approximation of one dimensional Biharmonic equations. Homotopy Perturbation and Homotopy Analysis methods were applied to solve initial value problems by Hashim et al., (2009). Taiwo (2013) applied two collocation methods for the solutions of integral equations by cubic spline. Mittal and Nigam (2008) solved integro-differential equations with Adomian Decomposition method. Recently, Khader et al. (2014) used the Chebyshev collocation method for solving fractional order Klein-Gordon equation. Bhrawy (2014) used a new Legendre collocation method for solving a two- dimensional fractional Riccati equation. the work of the above mentioned researchers motivated us to investigate this field of study. Here we proposed a Double Perturbation Collocation Method (DPCM) for the approximate solution of fractional Riccati equation.

DEFINITIONS OF RELEVANT TERMS

Definition 2.1:
Fractional derivatives refer to differential and integral equations that involve non-integer order. The general form is;

$$D^{\alpha} y(x) = f(x, y(x)) \tag{2.1}$$

subject to the conditions

$$D^{\alpha} y(0) = b_k \ , k = 0,1,2, \cdots \tag{2.2}$$

whereD^{α} represents the fractional derivative in the Caputo sense and $\alpha > 0$ is the order of the non-integer order derivative, and $f(x, y(x))$ is a continuous arbitrary smooth function. The b_k; $k = 0,1,2, \cdots$ are constants.

Definition 2.2:
Fractional Riccati differential Equation: this is of the form

$$D^{\alpha} y(t) = P(t) + Q(t)y(t) + R(t)y^2(t), \qquad t > 0, \qquad (0 < \alpha \leq 1 \tag{2.3}$$

subject to the initial condition:

$$y(0) = y_0 \tag{2.4}$$

where$P(t), Q(t)$ and $R(t)$ are real functions, also y_0 is a constant and α is a real number.

Definition 2.3:
A real function $f(x), x > 0 \in N$is said to be in space $C_{\alpha}, \alpha \in R$if there exist a real number $\rho > \alpha$, such that

$$f(x) = x\rho f_1(x) \tag{2.5}$$

where$f_1(x) = C[0,\infty]$. If $\beta \leq \alpha$, then$C_\alpha \in C_\beta$.

Definition 2.4:
A function $f(x), x > 0$ is said to be in space $C_m^\alpha, n \in N_0, N_0 = 1,2,\cdots$, if $f(n)$is in C_α.

Definition 2.5:
The gamma function is the generalization of the factorial for all positive real numbers. This is written as;

$$\Gamma(x) = \int_0^\alpha e^{-t} t^{x-1} dt \tag{2.6}$$

Definition 2.6:
The fractional derivative of $f(t)$ in the Caputo sense is defined as Podlubny (1999), Azizi and Loghmani (2013).

$$D_*^\alpha f(t) = \frac{1}{\Gamma(m-a)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau \tag{2.7}$$

for$m - 1 < \alpha \leq m, m \in N, t > 0$;
stated here are basic properties of Caputo derivatives if k is a constant, then;

$$D_*^\alpha f(k) = 0 \tag{2.8}$$

$$D_*^\alpha J^\alpha f(t) = f(t) \tag{2.9}$$

$$D_*^\alpha J^\alpha f(t) = f(t) - \sum_{j=0}^{m-1} f^{(j)}(0^+) \frac{t^j}{j!}, \qquad t > 0 \tag{2.10}$$

$$D_*^\alpha \big(k_1 f(t) + k_2 f(t)\big) = k_1 D_*^\alpha f(t) + k_2 D_*^\alpha f(t) \tag{2.11}$$

Chebyshev Polynomials

Here, basic properties of the Chebyshev polynomials needed for our work are stated.
The Chebyshev polynomials of the first kind and of degree k are defined on the interval $[-1,1]$ (see Snyder,1966) as;

$$T_k(t) = \cos^{-1}(k\cos(t)) \tag{3.1}$$

$$T_0(t) = 1, \qquad T_1(t) = t, \qquad T_2(t) = 2t^2 - 1. \tag{3.2}$$

and the recurrence relation is given as

$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t), \qquad k = 2,3,\cdots \tag{3.3}$$

The analytic form of the Chebyshev polynomial is:

$$T_k(t) = \sum_{j=0}^{p} (-1)^j\, 2^{k-2j-1} \frac{k(k-j-1)!}{j!\,(k-2j)!} t^{k-2j} \tag{3.4}$$

where$p = \frac{k}{2}$ is the integer part of $\frac{k}{2}$. The orthogonality condition is given by

$$\int_{-1}^{1} \frac{T_j(t)T_k(t)}{w(t)} dt = \begin{cases} \pi, & \text{for } j = k = 0; \\ \frac{\pi}{2}, & \text{for } j = k \neq 0; \\ 0, & \text{for } j \neq k. \end{cases} \tag{3.5}$$

The weight function

$$w(t) = \sqrt{1 - t^2} \tag{3.6}$$

Shifted Chebyshev Polynomials

Snyder(1966) stated shifted Chebyshev polynomials of degree n on the closed interval [0 ,L] as;

$$T_n^*(t) = T_n\left(\frac{2t}{L} - 1\right) \tag{3.7}$$

The recurrence formula on the closed interval [0,1]is;

$$T_{n+1}^*(t) = 2(2t - 1)T_n^*(t) - T_{n-1}^*(t) \tag{3.8}$$

Also,

$$T_0^*(t) = 1, \qquad T_1^*(t) = 2t - 1, \qquad T_2^*(t) = 8t^2 - 8t + 1 \tag{3.9}$$

Thus, the analytic form of the shifted Chebyshev polynomials is;

$$T_n^*(t) = \sum_{j=0}^{\infty} (-1)^{n-j} \frac{2^{2j} n(n + j - 1)!}{L^j (2j)!\,(n - j)!} \tag{3.10}$$

$n = 1,2,3,\cdots$. The function $y(t)$ in $L^2[0, L]$ i.e. $y(t)$ belongs to the space of square integrable in $[0, L]$ can be written in terms of shifted Chebyshev Polynomials as;

$$y(t) = \sum_{k=0}^{\infty} c_k T_k^*(t) \tag{3.11}$$

where$y(t)$ denotes the approximate solution of the given equation $c_k$is constant. $c_k$is defined for $k = 0,1,2,\cdots$ as:

$$c_0 = \frac{2}{\pi}\int_0^L y(t)T_0^*(t)w^*(t)dt \tag{3.12}$$

and

$$c_k = \frac{2}{\pi}\int_0^L y(t)T_k^*(t)w^*(t)dt \tag{3.13}$$

Where$w^*(t) = \frac{1}{\sqrt{Lt - t^2}}$. We consider only the first $(m + 1)$-terms of the shifted Chebyshev polynomials and the added perturbed part for the approximate solution of the given problem(2.3), so we write;

$$y_m(t) = \sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{\infty} \tau_k T_k^*(t) \tag{3.14}$$

Khader et al., (2014) proposed the formula below for the approximation of the fractional part of fractional derivative.

Theorem: Let $y(t)$ be approximated by the Chebyshev polynomials and also let > 0 , then,

$$D^\alpha y(t) = \sum_{j=[\alpha]}^{\infty} \sum_{k=[\alpha]}^{\infty} \left(c_j w_{j,k}^\alpha t^{k-\alpha} + \tau_j w_{j,k}^\alpha t^{k-\alpha}\right) \tag{3.15}$$

where,

$$w_{j,k}^{(\alpha)} = (-1)^{j-k} \frac{2^{2k} j(j + k - 1)!\,\Gamma(k + 1)}{L^k (j - k)!\,(2k)!\,\Gamma(k + 1 - \alpha)} \tag{3.16}$$

From the linear operator property of Caputo's fractional derivative, we can write

$$D^{\alpha}\left(y_m(t)\right)=\sum_{j=0}^{m} c_j D^{\alpha}\left(T_j^*(t)\right)+\sum_{j=0}^{m}\tau_j D\hat{}\alpha\left(T_j^*(t)\right) \quad (3.17)$$

$$D^{\alpha}\left(T_j^*(t)\right)=(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha} \quad (3.18)$$

METHODOLOGY

In this section, we present the Double Perturbation Collocation Method (DPCM) for the approximate solution of fractional Riccati differential equations. We consider Riccati fractional differential equation of the type in equation (2.3) with condition in equation (2.4). We use the trial double perturbation approximate solution

$$y_m(t)=\sum_{k=0}^{m} c_k T_k^*(t)+\sum_{k=0}^{m}\tau_k T_k^*(t) \quad (3.19)$$

where$c_0, c_1, c_2 \cdots, c_m$and $\tau_0, \tau_1, \tau_2 \cdots, \tau_m$ are constants. $T_k^*$are shifted Chebyshev Polynomials defined in equation (3.8).

In this method, two cases of equation (2.3) are considered.

Case I

$$D^{\alpha}y(t)=P(t)+Q(t)y(t) \quad (3.20)$$

Using equation (3.14) in equation (3.19), we have

$$D^{\alpha}\left[\sum_{k=0}^{m} c_k T_k^*(t)+\sum_{k=0}^{m}\tau_k T_k^*(t)\right]=Q(t)\left[\sum_{k=0}^{m} c_k T_k^*(t)+\sum_{k=0}^{m}\tau_k T_k^*(t)\right]+P(t) \quad (3.21)$$

$$D^{\alpha}\left[\sum_{k=0}^{m} c_k T_k^*(t)\right]+D^{\alpha}\left[\sum_{k=0}^{m}\tau_k T_k^*(t)\right] = Q(t)\left[\sum_{k=0}^{m} c_k T_k^*(t)\right]+Q(t)\left[\sum_{k=0}^{m}\tau_k T_k^*(t)\right]+P(t) \quad (3.22)$$

Using the properties of Caputo's derivative, we have;

$$\sum_{k=0}^{m} c_k D^{\alpha}T_k^*(t)+\sum_{k=0}^{m}\tau_k D^{\alpha}T_k^*(t) = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.23)$$

$$\sum_{j=1}^{m} c_k D^{\alpha}\left\{\sum_{k=1}^{j}(-1)^{j-k}\frac{2^{2k}j(j+k-1)!}{L^k(j-k)!\,(2k)!}t^k\right\} + \sum_{j=1}^{m}\tau_k D^{\alpha}\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!}{L^k(j-k)!\,(2k)!}t^k\right\} = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.24)$$

$$c_k \sum_{j=1}^{m}\sum_{k=1}^{j}\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!}{L^k(j-k)!\,(2k)!}D^{\alpha}t^{k}\right\} + \tau_k \sum_{j=1}^{m}\sum_{k=1}^{j}\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)}{L^k(j-k)!\,(2k)!}D^{\alpha}t^{k}\right\} = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.25)$$

$$\sum_{j=1}^{m}\sum_{k=1}^{j}c_k\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\} + \sum_{j=1}^{m}\sum_{k=1}^{j}\tau_k\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\} = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.26)$$

$$\sum_{j=1}^{m}\left\{c_1(-1)^{j-1}\frac{2^{2\cdot1}j(j+1-1)!\,\Gamma(1+1)}{L^1(j-1)!\,(2\cdot1)!\,\Gamma(1+1-\alpha)}t^{1-\alpha} + c_2(-1)^{j-2}\frac{2^{2\cdot2}j(j+2-1)!\,\Gamma(2+1)}{L^1(j-2)!\,(2\cdot2)!\,\Gamma(2+1-\alpha)}t^{2-\alpha} + \cdots + c_m(-1)^{j-m}\frac{2^{2\cdot m}j(j+m-1)!\,\Gamma(m+1)}{L^m(j-m)!\,(2\cdot m)!\,\Gamma(m+1-\alpha)}t^{m-\alpha}\right\} + \sum_{j=1}^{m}\left\{\tau_1(-1)^{j-1}\frac{2^{2\cdot1}j(j+1-1)!\,\Gamma(1+1)}{L^1(j-1)!\,(2\cdot1)!\,\Gamma(1+1-\alpha)}t^{1-\alpha} + \tau_2(-1)^{j-2}\frac{2^{2\cdot2}j(j+2-1)!\,\Gamma(2+1)}{L^1(j-2)!\,(2\cdot2)!\,\Gamma(2+1-\alpha)}t^{2-\alpha} + \cdots + \tau_m(-1)^{j-m}\frac{2^{2\cdot m}j(j+m-1)!\,\Gamma(m+1)}{L^m(j-m)!\,(2\cdot m)!\,\Gamma(m+1-\alpha)}t^{m-\alpha}\right\} = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.27)$$

$$\sum_{j=1}^{m}\{c_1a_1(j)t^{1-\alpha}+c_2a_2(j)t^{2-\alpha}+\cdots+c_ma_m(j)t^{m-\alpha}\} + \sum_{j=1}^{m}\{\tau_1a_1(j)t^{1-\alpha}+\tau_2a_2(j)t^{2-\alpha}+\cdots+\tau_ma_m(j)t^{m-\alpha}\} = Q(t)[c_0T_0^*(t)+c_1T_1^*(t)+\cdots+c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t)+\tau_1T_1^*(t)+\cdots+\tau_mT_m^*(t)]+P(t) \quad (3.28)$$

$$\begin{aligned} c_1 a_{11}(j)t^{1-\alpha} + c_2 a_{12}(j)t^{2-\alpha} + \cdots + c_m a_{1m}(j)t^{m-\alpha} + c_1 a_{21}(j)t^{1-\alpha} \\ + c_2 a_{22}(j)t^{2-\alpha} + \cdots + c_m a_{2m}(j)t^{m-\alpha} + \cdots + c_1 a_{m1}(j)t^{1-\alpha} \\ + c_2 a_{m2}(j)t^{2-\alpha} + \cdots + c_m a_{mm}(j)t^{m-\alpha} + \tau_1 a_{11}(j)t^{1-\alpha} \\ + \tau_2 a_{12}(j)t^{2-\alpha} + \cdots + \tau_m a_{1m}(j)t^{m-\alpha} + \tau_1 a_{21}(j)t^{1-\alpha} \\ + \tau_2 a_{22}(j)t^{2-\alpha} + \cdots + \tau_m a_{2m}(j)t^{m-\alpha} + \cdots + \tau_1 a_{m1}(j)t^{1-\alpha} \\ + \tau_2 a_{m2}(j)t^{2-\alpha} + \cdots + \tau_m a_{mm}(j)t^{m-\alpha} \\ = Q(t)[c_0 T_0^*(t) + c_1 T_1^*(t) + \cdots + c_m T_m^*(t)] \\ + Q(t)[\tau_0 T_0^*(t) + \tau_1 T_1^*(t) + \cdots + \tau_m T_m^*(t)] + P(t) \end{aligned} \quad (3.29)$$

$$\begin{aligned} -Q(t)T_0^*(t)c_0 + \Big((a_{11}(j) + a_{21}(j) + \cdots + a_{m1}(j)) - Q(t)T_1^*(t)\Big)c_1 t^{1-\alpha} \\ + \Big((a_{12}(j) + a_{22}(j) + \cdots + a_{m2}(j)) - Q(t)T_2^*(t)\Big)c_2 t^{2-\alpha} + \cdots \\ + \Big((a_{1m}(j) + a_{2m}(j) + \cdots + a_{mm}(j)) - Q(t)T_m^*(t)\Big)c_m t^{m-\alpha} \\ -Q(t)T_0^*(t)\tau_0 + \Big((a_{11}(j) + a_{21}(j) + \cdots + a_{m1}(j)) - Q(t)T_1^*(t)\Big)\tau_1 t^{1-\alpha} \\ + \Big((a_{12}(j) + a_{22}(j) + \cdots + a_{m2}(j)) - Q(t)T_2^*(t)\Big)\tau_2 t^{2-\alpha} + \cdots \\ + \Big((a_{1m}(j) + a_{2m}(j) + \cdots + a_{mm}(j)) - Q(t)T_m^*(t)\Big)\tau_m t^{m-\alpha} \\ - P(t) = 0 \end{aligned} \quad (3.30)$$

We then collocated equation (3.30) at $t = t_r$ where

$$t_r = a + \left(\frac{b-a}{m-1}\right)r, \qquad r = 0,1,2,\cdots,m-2 \quad (3.31)$$

to get

$$\begin{aligned} -Q(t_r)T_0^*(t_r)c_0 + \Big((a_{11}(j) + a_{21}(j) + \cdots + a_{m1}(j))t^{1-\alpha} - Q(t_r)T_1^*(t_r)\Big)c_1 \\ + \Big((a_{12}(j) + a_{22}(j) + \cdots + a_{m2}(j))t^{2-\alpha} - Q(t_r)T_2^*(t_r)\Big)c_2 + \cdots \\ + \Big((a_{1m}(j) + a_{2m}(j) + \cdots + a_{mm}(j))t^{m-\alpha} - Q(t_r)T_m^*(t_r)\Big)c_m \\ -Q(t_r)T_0^*(t_r)\tau_0 + \Big((a_{11}(j) + a_{21}(j) + \cdots + a_{m1}(j))t^{1-\alpha} - Q(t_r)T_1^*(t_r)\Big)\tau_1 \\ + \Big((a_{12}(j) + a_{22}(j) + \cdots + a_{m2}(j))t^{2-\alpha} - Q(t_r)T_2^*(t_r)\Big)\tau_2 + \cdots \\ + \Big((a_{1m}(j) + a_{2m}(j) + \cdots + a_{mm}(j))t^{m-\alpha} - Q(t_r)T_m^*(t_r)\Big)\tau_m \\ - P(t_r) = 0 \end{aligned} \quad (3.32)$$

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} & \tau_{11} & \tau_{12} & \cdots & \tau_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} & \tau_{21} & \tau_{22} & \cdots & \tau_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} & \tau_{m1} & \tau_{m2} & \cdots & \tau_{mm} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_m \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \\ \vdots \\ \vdots \\ \vdots \\ B_{m+1} \end{pmatrix}$$

$$A_{11} = -Q(t_1)T_0^*(t_1)c_0,$$

$$A_{12} = \Big((a_{11}(1) + a_{21}(1) + \cdots + a_{m1}(1))t^{1-\alpha} - Q(t_1)T_1^*(t_1)\Big),$$

$$A_{1m} = \Big((a_{1m}(1) + a_{2m}(1) + \cdots + a_{mm}(1))t^{m-\alpha} - Q(t_1)T_m^*(t_1)\Big)$$

$$A_{21} = -Q(t_2)T_0^*(t_2)c_0,$$

$$A_{22} = \Big((a_{11}(2) + a_{21}(2) + \cdots + a_{m1}(2))t^{2-\alpha} - Q(t_2)T_2^*(t_2)\Big)$$

$$A_{2m} = \Big((a_{1m}(2) + a_{2m}(2) + \cdots + a_{mm}(2))t^{m-\alpha} - Q(t_2)T_2^*(t_2)\Big)$$

$$\vdots\vdots$$

$$A_{m1} = -\,Q(t_m)T_0^*(t_m)c_0,$$
$$A_{m2} = \left((a_{11}(m) + a_{21}(m) + \cdots + a_{m1}(m))t^{1-\alpha} - \,Q(t_m)T_1^*(t_m)\right)$$
$$A_{mm} = \left((a_{1m}(m) + a_{2m}(m) + \cdots + a_{mm}(m))t^{m-\alpha} - \,Q(t_m)T_m^*(t_m)\right)$$
$$\tau_{11} = -a_{11}(1)t_1^{1-\alpha}, \qquad \tau_{12} = -a_{22}(1)t_1^{2-\alpha},$$
$$\tau_{13} = -a_{33}(1)t_1^{3-\alpha}, \qquad \tau_{14} = -a_{m\alpha}(1)$$
$$\tau_{21} = -a_{11}(2)t_2^{1-\alpha}, \qquad \tau_{22} = -a_{22}(2)t_2^{2-\alpha},$$
$$\tau_{23} = -a_{33}(2)t_2^{3-\alpha}, \qquad \tau_{24} = -a_{m\alpha}(2)$$
$$\vdots$$
$$\tau_{m1} = -a_{11}(m)t_m^{1-\alpha}, \qquad \tau_{m2} = -a_{22}(m)t_m^{2-\alpha},$$
$$\tau_{m3} = -a_{33}(1)t_m^{3-\alpha}, \qquad \tau_{m4} = -a_{m\alpha}(m)$$

Case II

Substituting equations (3.14), and (3.15) into equation (2.3) we have

$$D^\alpha\left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{m} \tau_k T_k^*(t)\right] = P(t) + Q(t)\left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{m} \tau_k T_k^*(t)\right] + R(t)\left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{m} \tau_k T_k^*(t)\right]^2$$

$$D^\alpha\left[\sum_{k=0}^{m} c_k T_k^*(t)\right] + D^\alpha\left[\sum_{k=0}^{m} \tau_k T_k^*(t)\right] = P(t) + Q(t)\left[\sum_{k=0}^{m} c_k T_k^*(t)\right] + Q(t)\left[\sum_{k=0}^{m} \tau_k T_k^*(t)\right] + R(t)\left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{m} \tau_k T_k^*(t)\right]^2 \quad (3.33)$$

$$\sum_{k=0}^{m}\left\{\sum_{j=0}^{m}\sum_{k=0}^{j} c_k w_{j,k}^{(\alpha)} t^{k-\alpha}\right\} + \sum_{k=1}^{m}\left\{\sum_{j=0}^{m}\sum_{k=0}^{j} \tau_k w_{j,k}^{(\alpha)} t^{k-\alpha}\right\} = P(t) + Q(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)] + R(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t) + \tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]^2 \quad (3.34)$$

$$\sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} c_k w_{j,k}^{(\alpha)} t^{k-\alpha} + \sum_{k=1}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} \tau_k w_{j,k}^{(\alpha)} t^{k-\alpha} = P(t) + Q(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)] + Q(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)] + R(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t) + \tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]^2 + 2R(t)\left[\left(c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)\right)\left(\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)\right)\right] + R(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]^2 \quad (3.35)$$

$$\sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} c_k \left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\}$$
$$+\sum_{k=1}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} \tau_k \left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\}$$
$$= P(t) + Q(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)] \quad (3.36)$$
$$+ Q(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]$$
$$+ R(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t) + \tau_0T_0^*(t) + \tau_1T_1^*(t)$$
$$+ \cdots + \tau_mT_m^*(t)]^2$$
$$+ 2R(t)\left[\left(c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)\right)\left(\tau_0T_0^*(t)\right.\right.$$
$$\left.\left.+ \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)\right)\right]$$
$$+ R(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]^2$$

$$\sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} c_k \left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\}$$
$$+\sum_{k=1}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} \tau_k \left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t^{k-\alpha}\right\}$$
$$= P(t) + Q(t)[c_0T_0^*(t) + c_1T_1^*(t) + \cdots + c_mT_m^*(t)]$$
$$+ Q(t)[\tau_0T_0^*(t) + \tau_1T_1^*(t) + \cdots + \tau_mT_m^*(t)]$$
$$+ R(t)[c_0^2T_0^{*2}(t) + 2c_0c_1T_0^*(t)T_1^*(t) + 2c_0c_3T_0^*(t)T_3^*(t) + \cdots \quad (3.37)$$
$$+ c_1^2T_1^{*2}(t) + 2c_1c_2T_1^*(t)T_2^*(t) + 2c_3c_4T_3^*(t)T_4^*(t) + \cdots$$
$$+ 2c_{n-1}c_nT_{n-1}^*(t)T_n^*(t) + \cdots + c_n^2T_n^{*2}(t)]$$
$$+ R(t)[\tau_0^2T_0^{*2}(t) + 2\tau_0\tau_1T_0^*(t)T_1^*(t) + 2\tau_0\tau_3T_0^*(t)T_3^*(t) + \cdots$$
$$+ \tau_1^2T_1^{*2}(t) + 2\tau_1\tau_2T_1^*(t)T_2^*(t) + 2\tau_3\tau_4T_3^*(t)T_4^*(t) + \cdots$$
$$+ 2\tau_{n-1}\tau_nT_{n-1}^*(t)T_n^*(t) + \cdots + \tau_n^2T_n^{*2}(t)]$$
$$+ 2(t)\left[\left(c_0T_0^*(t)\cdot\tau_0T_0^*(t) + c_1T_1^*(t)\cdot\tau_1T_1^*(t) + \cdots + c_nT_n^*(t)\right.\right.$$
$$\left.\left.\cdot\,\tau_nT_n^*(t)\right)\right], \qquad 1 \le n \le m$$

We then collocated equation (3.37) at $t = t_r$ where

$$t_r = a + \left(\frac{b-a}{m-1}\right)r, \qquad r = 0,1,2,\cdots,m-2$$

to get

$$\sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} c_k\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t_r^{k-\alpha}\right\}$$
$$\begin{aligned}&+\sum_{k=1}^{m}\sum_{j=0}^{m}\sum_{k=0}^{j} \tau_k\left\{(-1)^{j-k}\frac{2^{2k}j(j+k-1)!\,\Gamma(k+1)}{L^k(j-k)!\,(2k)!\,\Gamma(k+1-\alpha)}t_r^{k-\alpha}\right\}\\
&= P(t_r)+Q(t_r)[c_0T_0^*(t_r)+c_1T_1^*(t_r)+\cdots+c_mT_m^*(t_r)]\\
&+Q(t_r)[\tau_0T_0^*(t_r)+\tau_1T_1^*(t_r)+\cdots+\tau_mT_m^*(t_r)]\\
&+R(t_r)[c_0^2T_0^{*2}(t_r)+2c_0c_1T_0^*(t)T_1^*(t_r)+2c_0c_3T_0^*(t_r)T_3^*(t_r)+\cdots\\
&+c_1^2T_1^{*2}(t_r)+2c_1c_2T_1^*(t_r)T_2^*(t)+2c_3c_4T_3^*(t_r)T_4^*(t_r)+\cdots\\
&+2c_{n-1}c_nT_{n-1}^*(t_r)T_n^*(t_r)+\cdots+c_n^2T_n^{*2}(t_r)]\\
&+R(t_r)[\tau_0^2T_0^{*2}(t_r)+2\tau_0\tau_1T_0^*(t_r)T_1^*(t_r)+2\tau_0\tau_3T_0^*(t_r)T_3^*(t_r)+\cdots\\
&+\tau_1^2T_1^{*2}(t_r)+2\tau_1\tau_2T_1^*(t_r)T_2^*(t_r)+2\tau_3\tau_4T_3^*(t_r)T_4^*(t_r)+\cdots\\
&+2\tau_{n-1}\tau_nT_{n-1}^*(t_r)T_n^*(t_r)+\cdots+\tau_n^2T_n^{*2}(t_r)]\\
&+2(t_r)\big[\big(c_0T_0^*(t_r)\cdot\tau_0T_0^*(t_r)+c_1T_1^*(t_r)\cdot\tau_1T_1^*(t_r)+\cdots+c_nT_n^*(t_r)\\
&\cdot\tau_nT_n^*(t_r)\big)\big],\qquad 1\le n\le m\end{aligned}\tag{3.38}$$

Similarly, equation (3.38) is put into matrix form as in the case of its linear counterpart.

$$\begin{pmatrix} A_{11}^* & A_{12}^* & \cdots & A_{1m}^* & \tau_{11}^* & \tau_{12}^* & \cdots & \tau_{1m}^*\\ A_{21}^* & A_{22}^* & \cdots & A_{2m}^* & \tau_{21}^* & \tau_{22}^* & \cdots & \tau_{2m}^*\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ A_{m1}^* & A_{m2}^* & \cdots & A_{mm}^* & \tau_{m1}^* & \tau_{m2}^* & \cdots & \tau_{mm}^* \end{pmatrix}\begin{pmatrix} c_0\\ c_1\\ \vdots\\ c_m\\ \tau_1\\ \tau_2\\ \vdots\\ \tau_m\end{pmatrix}=\begin{pmatrix} B_1^*\\ B_2^*\\ \vdots\\ B_m^*\\ \vdots\\ \vdots\\ \vdots\\ B_{m+1}^*\end{pmatrix}$$

4.0: NUMERICAL EXAMPLES BASED ON DOUBLE PERTURBATION COLLOCATION METHOD (DPCM)

Three numerical examples are presented to demonstrate the accuracy, effectiveness and efficiency of the proposed method.

Numerical Example 1

Consider the linear fractional differential equation (Abdulaziz et, al(2008)).

$$D^\alpha y(t) = y(t), t > 0, (0 < \alpha \le 2) \tag{4.1}$$
$$y(0) = 1, y\,(0) = -1 \tag{4.2}$$

the second condition is for $\alpha > 0$ the exact solution is $y(t) = e^{-t}$

Substituting equations (3.14) and (3.15) into equation (4.1) to get

$$\sum_{j=1}^{m}\sum_{k=1}^{j} c_j w_{j,k}^{(\alpha)} t^{k-\alpha} + \sum_{j=1}^{m}\sum_{k=1}^{j} \tau_j w_{j,k}^{(\alpha)} t^{k-\alpha} - \sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{m} \tau_k T_k^*(t) = 0 \tag{4.3}$$

We consider the case m = 5 in (4.3) above for the approximation:

$$\sum_{j=1}^{5}\sum_{k=1}^{j} c_j w_{j,k}^{(\alpha)} t^{k-\alpha} + \sum_{j=1}^{5}\sum_{k=1}^{j} \tau_j w_{j,k}^{(\alpha)} t^{k-\alpha} - \sum_{k=0}^{5} c_k T_k^*(t) + \sum_{k=0}^{5} \tau_k T_k^*(t) = 0 \tag{4.4}$$

We expand and simplify equation (4.4) to get:

$$-c_1 + c_2 - c_3 + c_4 - c_5 + \tau_1 - \tau_2 + \tau_3 - \tau_4 + \tau_5 - 8c_2t + 1c_3t + 32c_3t^3 - 400c_5t^2 + 1120c_5t^3 - 1280c_5t^4 + 2c_1t + 8c_2t^2 - 48c_3t^2 - 32c_4t + 160c_4t^2 - 256c_4t^3 + 50c_5t + c_0 + 128c_4t^4 + 512c_5t^5 - \tau_0 - 2\tau_1t - 8\tau_2t^2 + 8\tau_2t - 32\tau_3t^3 + 48\tau_3t^2 - 18\tau_3t - 128\tau_4t^4 + 256\tau_4t^3 - 160\tau_4t^2 + 32\tau_4t - 512\tau_5t^5 + 1280\tau_5t^4 - 1120\tau_5t^3 + 400\tau_5t^2 - 50\tau_5t \tag{4.5}$$

Substituting the initial condition into the approximate solution in (3.31) gives;

$$-c_1 + c_2 - c_3 + c_4 - c_5 + c_0 - \tau_0 + \tau_1 - \tau_2 + \tau_3 - \tau_4 + \tau_5 = 1 \tag{4.6}$$

$$x_i = \frac{i}{11}, \qquad i = 1,2,3,\cdots,11 \tag{4.7}$$

We then collocated (4.5) using the values of $x_i$defined by to get eleven more equations, these twelve equations above were solved implementing Gaussian elimination method to obtain the values for the twelve unknown constants which were then substituted into the perturbed approximate solution in (3.14) to get;

$$y(t) = 1.000000 - 0.9999214570t + 0.4994227054t^2 - 0.1647069987t^3 + 0.003817244034t^4 - 0.005089658679t^5 \tag{4.19}$$

Similarly, taking$m = 6$, we obtained fourteen equations in fourteen unknowns: These equations were then solved with Gaussian elimination method to obtain the values unknown constants which were equally substituted into (2.14) to have the approximate solution as;

$$y(t) = 0.9999999999 - 0.9999948133t + 0.4999529330t^2 - 0.1664599129t^3 + 0.04116099974t^4 - 0.007626895554t^5 + 0.0008479342078t^6 \tag{4.20}$$

These numerical results for $m = 5$ and $m = 6$ when evaluated at some equidistant points, were compared with the exact solutions in Tables 1 and 2 below.

Table 1: Numerical Results and Errors for Example 1 for case m =5

t	Exact	Approximate	Error
0.0	1.0000000000	1.0000000000	0.00000000
0.1	0.9048411407	0.9048411407	0.00000000
0.2	0.8187344080	0.8187344080	0.00000000
0.3	0.7408213463	0.7408213463	0.00000000
0.4	0.6703228986	0.6703228986	0.00000000
0.5	0.6065332988	0.6065332988	0.00000000
0.6	0.5488139644	0.5488139644	0.00000000
0.7	0.4965873892	0.4965873892	0.00000000
0.8	0.4493310348	0.4493310348	0.00000000
0.9	0.4065712235	0.4065712235	0.00000000

1.0	0.3678770313	0.3678770313	0.00000000

Table 2: Numerical Results and Errors for Example 1 for case m=6

t	Exact	Approximate	Error
0.0	1.0000000000	0.9999999999	1.0000E-10
0.1	0.9048411407	0.9048411407	0.00000000
0.2	0.8187344080	0.8187344080	0.00000000
0.3	0.7408213463	0.7408213463	0.00000000
0.4	0.6703228986	0.6703228986	0.00000000
0.5	0.6065332988	0.6065332988	0.00000000
0.6	0.5488139644	0.5488139644	0.00000000
0.7	0.4965873892	0.4965873892	0.00000000
0.8	0.4493310348	0.4493310348	0.00000000
0.9	0.4065712235	0.4065712235	0.00000000
1.0	0.3678770313	0.3678770313	0.00000000

Numerical Example 2

Consider the fractional nonlinear Riccati differential equation (Khader et al,(2014)).

$$D^{\alpha}y(t) + y^2(t) - 1 = 0, \qquad t > 0, \qquad \alpha \in (0,1] \tag{4.21}$$

Subject to the initial condition;

$$y(0) = 0$$

when$\alpha = 1$,we get the Riccati differential equation with the exact solution

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \tag{4.22}$$

To solve the problem (4.21) using Newton linearization method we proceed as follows. We approximate the function $y_m(t)$ at m=5, using equations (3.14) and (3.15).

$$\sum_{j=1}^{5}\sum_{k=1}^{j} c_j w_{j,k}^{(\alpha)} t^{k-\alpha} + \sum_{j=1}^{5}\sum_{k=1}^{j} \tau_j w_{j,k}^{(\alpha)} t^{k-\alpha} - \left(\sum_{k=0}^{5} c_k T_k^*(t) + \sum_{k=0}^{5} \tau_k T_k^*(t)\right)^2 - 1 = 0 \tag{4.23}$$

Equation (4.23) is expanded using a modified program written in Maple 13 software. Also, the initial condition is substituted into (3.31) to have;

$$\sum_{k=0}^{5} c_k T_k^*(0) + \sum_{k=0}^{5} \tau_k T_k^*(0) = 0 \tag{4.25}$$

We then collocated equation (4.24) at points

$$x_i = \frac{i}{11}, \qquad i = 1,2,3,\cdots,11 \tag{4.26}$$

Equation (4.25) together with equation (4.26) constituted twelve systems of nonlinear equations with twelve unknown constants which are then solved for the twelve unknowns constants using Newton's linearization scheme given below:

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$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+2} \\ \vdots \\ x_{n,n+n} \end{pmatrix} = \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{n,n} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_{1,n}, x_{2,n}, \cdots x_{n,n}) \\ f_2(x_{1,n}, x_{2,n}, \cdots x_{n,n}) \\ \vdots \\ f_n(x_{1,n}, x_{2,n}, \cdots x_{n,n}) \end{pmatrix} \quad (4.27)$$

This iterative procedure converges at the fifth iterations. Constants obtained were then substituted into equation (3.14) to obtain the approximate solution:

$$\begin{aligned} y_5(t) = {} & 1.0050 \times 10^{-10} + 0.9993788403t + 0.0156030431t^2 \\ & - 0.04181689820t^3 + 0.1793761769t^4 \\ & + 0.01457934619t^5 \end{aligned} \quad (4.28)$$

Also for $m = 6$ in equation (4.21), we obtained fourteen linear equations containing fourteen unknown constants which were also solved with Newton linearization scheme given in equation (4.27). The approximate solution for case $m = 6$ is given as (4.29).

$$\begin{aligned} y_6(t) = {} & 2.904 \times 10^{-11} + 1.000008670t - 0.00023248899t^2 \\ & - 0.3308565625t^3 - 0.01327514300t^4 + 0.173842917t^5 \\ & - 0.006854730555t^6 \end{aligned} \quad (4.29)$$

Results obtained after evaluated at some equidistant points are tabulated below: We noted here that the results are not available for $0 \leq \alpha < 1$.

Table 3: Numerical Results and Errors for Example 2 for case m=5

t	Exact	Approximate	Error
0.0	0.0000000000	1.005000E-10	1.00500E-10
0.1	0.0996679945	0.0996935374	2.55429E-05
0.2	0.1973753203	0.1974368745	6.15542E-05
0.3	0.2913126124	0.2913448828	3.22704E-05
0.4	0.3799489622	0.3799279458	2.10164E-05
0.5	0.4621171573	0.4620744649	4.26924E-05
0.6	0.5370495670	0.5370333622	6.20480E-05
0.7	0.6043677771	0.6043965880	2.88109E-05
0.8	0.6640367702	0.6640816230	4.48528E-05
0.9	0.7162978702	0.7163139850	1.61148E-05
1.0	0.7615941560	0.7616097327	1.55767E-05

Table 4: Numerical Results and Errors for Example 2 for case m=6

t	Exact	Approximate	Error
0.0	0.0000000000	2.904000E-11	2.904000E-11
0.1	0.0996679945	0.0996680281	3.350000E-08
0.2	0.1973753203	0.1973755881	2.678000E-07
0.3	0.2913126124	0.2913135160	9.036000E-07
0.4	0.3799489622	0.3799511042	2.142000E-06
0.5	0.4621171573	0.4621213411	4.183800E-06
0.6	0.5370495670	0.5370567966	7.229600E-06
0.7	0.6043677771	0.6043761555	8.378400E-06
0.8	0.6640367702	0.6640213938	1.537640E-05
0.9	0.7162978702	0.7161456036	1.572666E-05
1.0	0.7615941560	0.7609514616	1.926944E-05

Numerical Example 3

Consider the fractional Riccati differential equation (Khader et al,(2014)).

$$D^{\alpha}y(t) - y(t)^2 - t^2 = 0, \qquad \alpha \in (0,1], \qquad t > 0, \qquad y(0) = 1 \tag{4.30}$$

The exact solution for the problem at $\alpha = 1$ is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \left(\frac{1}{2}\right)\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{4.31}$$

To solve the equation above using DPCM, we substitute equations (3.14) and (3.15) into equation (4.30) to get

$$\sum_{j=1}^{5}\sum_{k=1}^{j} c_j w_{j,k}^{(\alpha)} t^{k-\alpha} + \sum_{j=1}^{5}\sum_{k=1}^{j} \tau_j w_{j,k}^{(\alpha)} t^{k-\alpha} - \left(\sum_{k=0}^{5} c_k T_k^*(t) + \sum_{k=0}^{5} \tau_k T_k^*(t)\right) - t^2 = 0 \tag{4.32}$$

We proceeded as before by expanding equation (4.32) and then collocating the slightly perturb resulting equation. This leads to eleven non-linear ordinary differential equations while the twelve equation is given by substituting the initial condition to the approximate solution equation (3.14). The approximate results is:

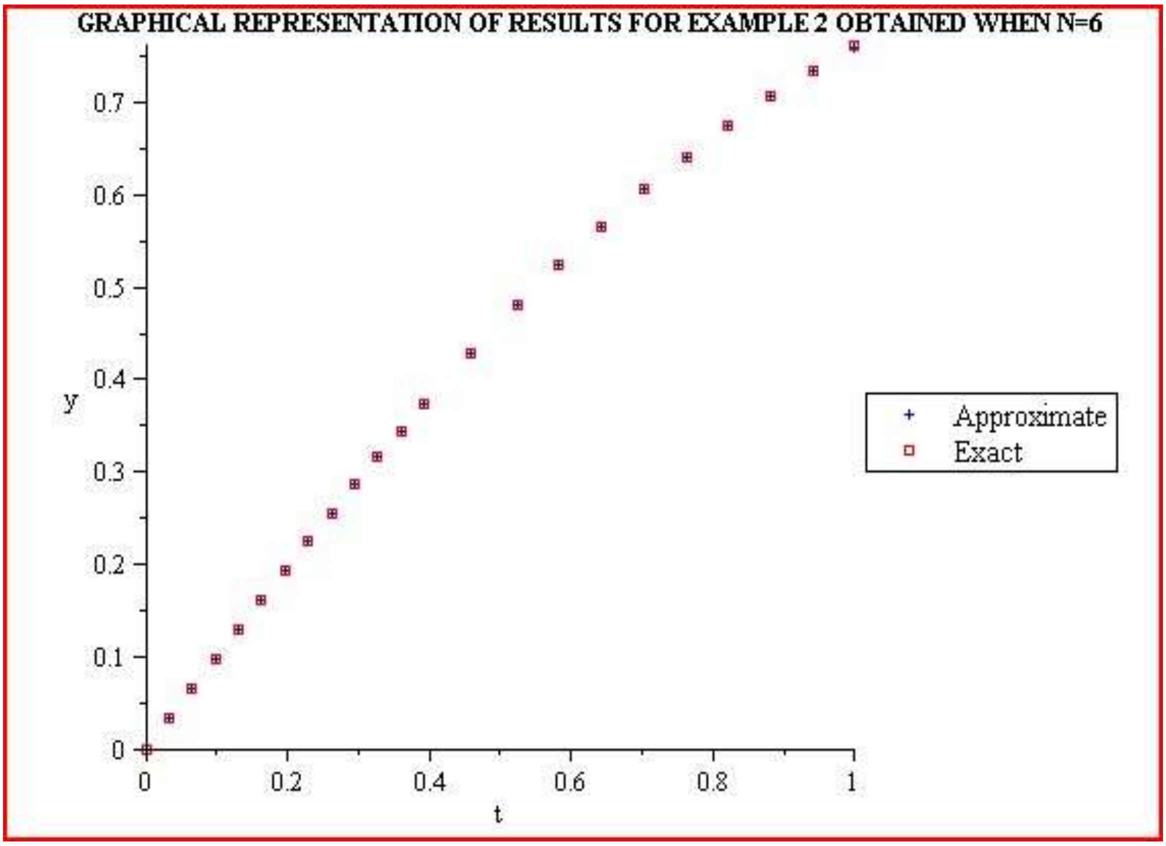
$$y_5(t) = 1.006873741t + .8338272666t^2 + 1.173327776t^3 - 1.952928176t^4 + .6281447803t^5 \tag{4.33}$$

The results generated are tabulated in Table 5.

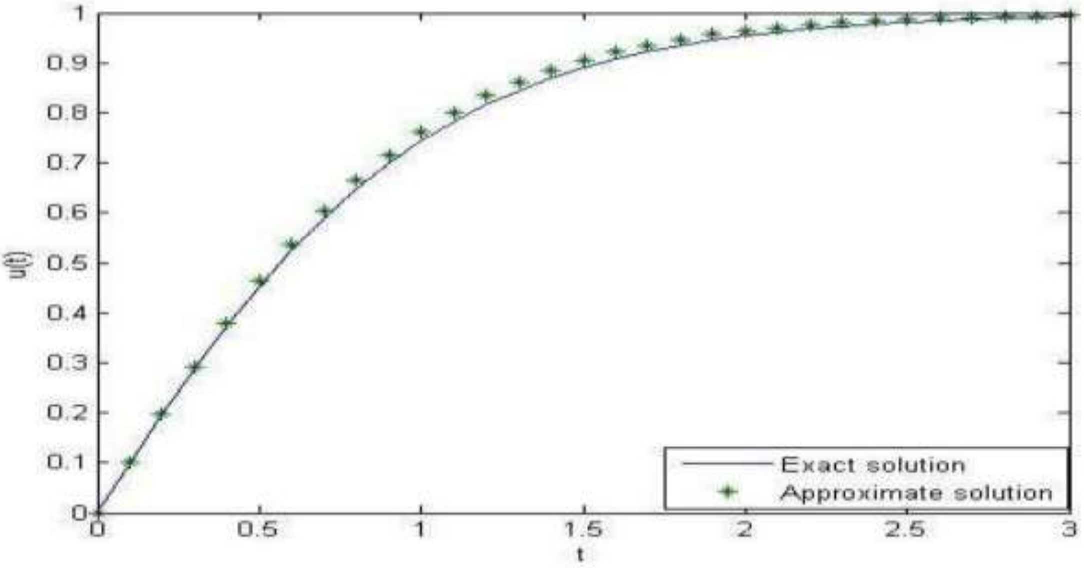
Table 5: Numerical Results and Errors for Example 3 for case m=5

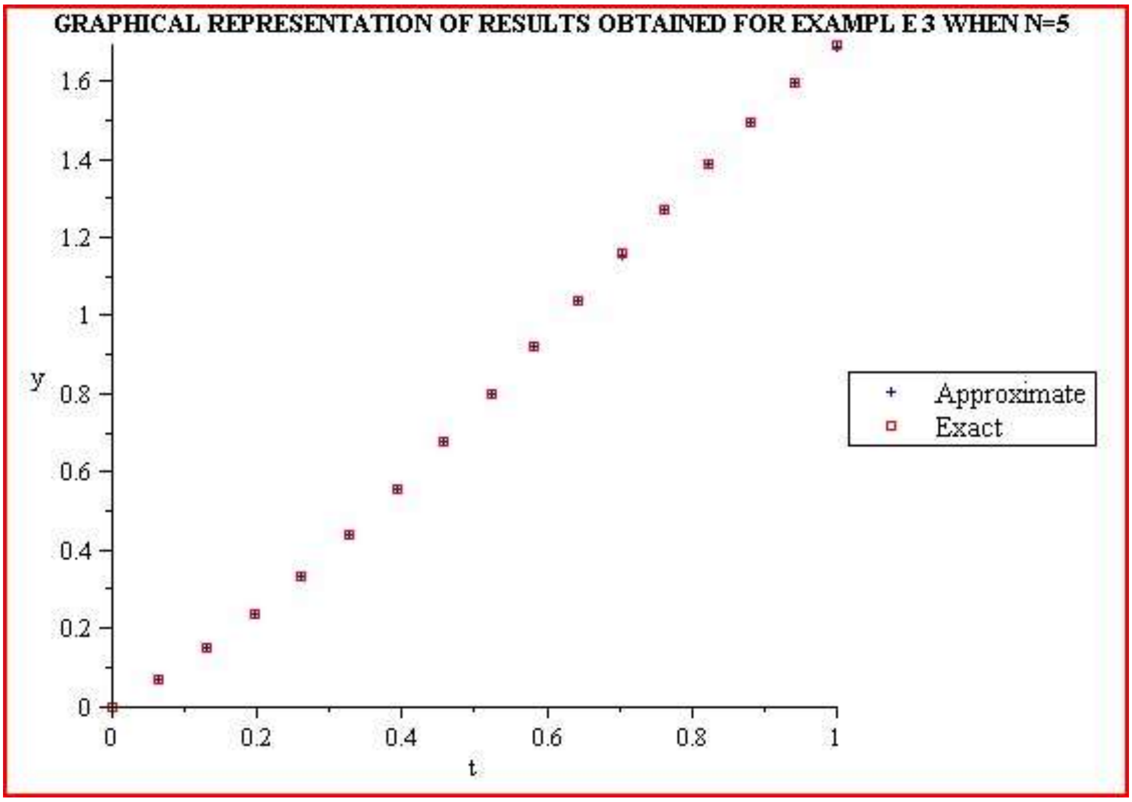
t	Exact	Approximate	Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.1102951967	0.1100099632	2.852335E-04
0.2	0.2419767994	0.2411907823	7.860171E-04
0.3	0.3951048483	0.3944940998	6.107485E-04
0.4	0.5678121658	0.5676920780	1.200878E-04
0.5	0.7560143927	0.7561311725	1.167798E-04
0.6	0.9535662156	0.9534859065	8.030910E-05
0.7	1.1529489660	1.1525126440	4.363220E-04
0.8	1.3463636550	1.3458033650	5.602900E-04
0.9	1.5269113130	1.5265394370	3.718760E-04
1.0	1.6894983900	1.6892453880	2.530020E-04

GRAPHICAL REPRESENTATION OF RESULTS

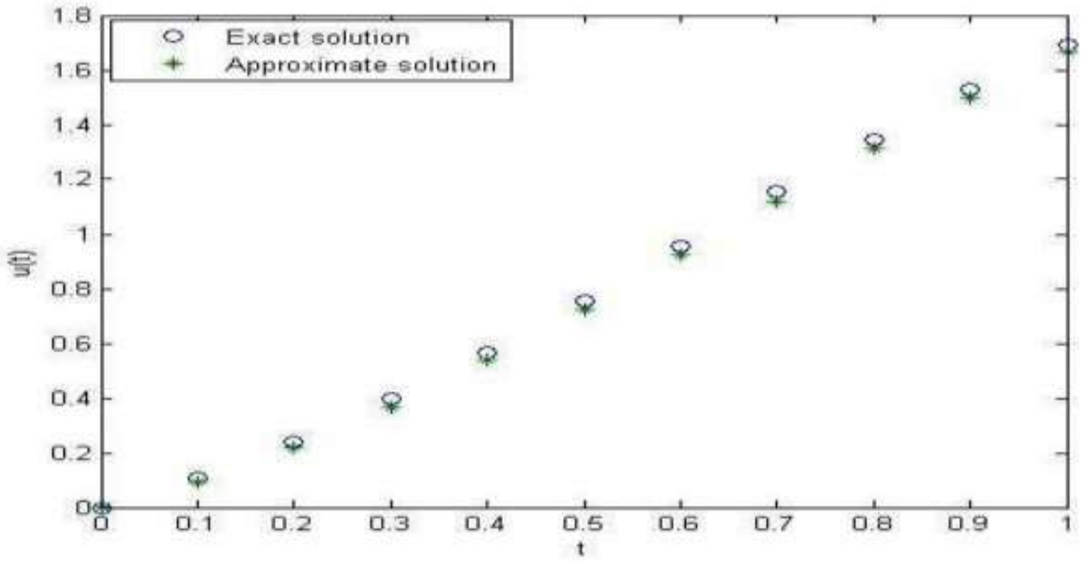


Graphical representation of results, Sweilamet. al,(2012)





Graphical representation of results,:Khader et. al. (2014)



CONCLUSION

In this work, we have applied Double Perturbation Collocation Method (DPCM) to solve fractional Riccati differential equations with the approximate solution assumed in equation (3.19). We solved both linear and nonlinear examples and compared our results with the exact solution for case $\alpha = 1$. For these, we found that the proposed method produced very good results. The tables and graphical representation of results revealed the accuracy of the method. For linear cases considered, the approximate solutions coincided with the exact solutions. Also for the nonlinear examples, the results obtained agreed with the results obtained in the literature. The results also show, that the new method is performing better as m is increasing.

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