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DEGREE OF BEST ONE-SIDED INTERPOLATION BY HERMITE-FEJER POLYNOMIALS IN WEIGHTED SPACES

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ABSTRACT: *In this search we find a degree of best one-sided approximation of the function f that lies in weighted space ($L_{p,\alpha}$-space) by interpolation-operators which based on Hermite-Fejer interpolation polynomial constructed on the zeroes of Stieltjes polynomial $E_{n+1}^{(\lambda)}$ and the product $E_{n+1}^{(\lambda)}P_n^{(\lambda)}$ for $0 \le \lambda \le 1$ and $0 \le \lambda \le \frac{1}{2}$ (resp.). Here we denoted of these interpolation-operators by $H_{n+1}^{*,\pm}(f,x_n)$ and $H_{2n+1}^{**,\pm}(f,x_n)$ where x_n is the set of zeroes of $E_{n+1}^{(\lambda)}$ and $E_{n+1}^{(\lambda)}P_n^{(\lambda)}$ (resp.) such that $P_n^{(\lambda)}$ is ultra-spherical polynomial with respect to $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$.The result which we end in it that the limit of differences between $H_{n+1}^{*,+}$ and $H_{n+1}^{*,-}$ is zero and $H_{2n+1}^{**,+}, H_{2n+1}^{**,-}$(resp.)i.e. $(\underbrace{\lim}_{n\to\infty}(H_{n+1}^{*,+} - H_{n+1}^{*,-}) = 0)$(resp. for $H_{2n+1}^{**,\pm}(f,x_n)$).Also in this search we shall prove inverse theorem by using equivalent result between $E_n(f)$ and $\tilde{E}_n(f)$ and inverse theorem in a best approximation case such that both pervious theorems are in weighted space where the weight function is a generalized Jacobi (GL) weighted $u(x)$. Finally we try to estimate degree of best one-sided approximation of the derivative of the function f in weighted space*

KEYWORDS: Interpolation, Hermite-Fejer, Polynomials, Weighted Spaces

INTRODUCTION

Throughout this paper, we use the weight function:
$u_\alpha(x) = \prod_{k=0}^{r}|t_k - x|^{\alpha_k}, \alpha_k \ge -1, -1 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1, |x| \le 1$ Where t_r is any partition for $[-1,1]$. For $1 \le p \le \infty$ the weighted space is define by:
$L_{p,\alpha=}\{f | f: X \longrightarrow R \text{ , such that } |f(x)u_\alpha(x)| \le M, \alpha \ge 1\}$ Such that for the function$f$, $\|f\|_{p,\alpha} = \left[\int_a^b |f(x)u_\alpha(x)|^p\, dx\right]^{\frac{1}{p}} < \infty$ also:
$\|f\|_{\infty,\alpha} = \{sup\{|f(x)u_\alpha(x)|, x \in X\}\} < \infty$, $X = [-1,1]$.In **[1]** Stieltjes polynomial $E_{n+1}^{(\lambda)}$with respect to is satisfying:
$\int_{-1}^{1}\omega_\lambda(x)P_n^{(\lambda)}(x)E_{n+1}^{(\lambda)}(x)x^m dx \begin{cases} = 0, 0 \le m < n+1 \\ \ne 0, \quad m = n+1. \end{cases}$
And denoted by $H_{n+1}[f,x]$ *and* $H_{2n+1}[f,x]$ to the Hermite –Fejer interpolation polynomials based on the zeros of the Stielties polynomials $E_{n+1}^{(\lambda)}$ *and* $E_{n+1}^{(\lambda)}P_n^{(\lambda)}$ (resp.) for $0 \le \lambda \le 1$ *and* $0 \le \lambda \le \frac{1}{2}$ (resp.) and $X = [-1,1]$ with respect to the partition $x_n = \{x_{1n}, x_{2n}, \dots \dots \dots x_{nn}\}, n \ge 1$ be a pairwise distinct nodes. We denoted to the zeroes of $P_n^{(\lambda)}$ and $E_{v,n}^{(\lambda)}$ by $x_{v,n}^{(\lambda)} = cos\emptyset_{v,n}^{(\lambda)}, v = 1, \dots, n$ and $\xi_{\mu,n+1}^{(\lambda)} = cos\theta_{\mu,n+1}^{(\lambda)}, \mu = 1, \dots, n+1$ and we denoted the zeroes of:
$F_{n+1}^{(\lambda)} = P_n^{(\lambda)}E_{n+1}^{(\lambda)}$ By $y_{v,2n+1}^{(\lambda)} = cos\psi_{v,2n+1}^{(\lambda)}, v = 1, \dots, 2n+1,$

Also we set $\varphi(x) = \sqrt{1-x^2}$ .The Hermite-Fejer interpolation $H_{n+1}[f,x]$ of f with respect to the zeros of $E_{n+1}^{(\lambda)}(x)$is admits representation:

$$H_{n+1}[f](x) = \sum_{k=1}^{2n+1} f\left(\xi_{u,n+1}^{(\lambda)}\right)[1 - \frac{\acute{E}_{n+1}^{(\lambda)}\left(\xi_{u,n+1}^{(\lambda)}\right)}{E_{v,n}^{(\lambda)}(\xi_{u,n+1}^{(\lambda)})}(x - \xi_{u,n+1}^{(\lambda)})]L_{k,n+1}^2(x).$$

The Hermite-Fejer interpolation $H_{2n+1}[f,x]$ of f with respect to the zeros of $F_{2n+1}^{(\lambda)}(x)$is admits representation:

$$H_{2n+1}[f](x) = \sum_{v=1}^{2n+1} f\left(y_{v,2n+1}^{(\lambda)}\right)\left[1 - \frac{\acute{F}_{n+1}^{(\lambda)}\left(y_{v,2n+1}^{(\lambda)}\right)}{\acute{F}_{v,n}^{(\lambda)}(y_{v,n+1}^{(\lambda)})}\left(x - y_{v,n+1}^{(\lambda)}\right)\right] L_{v,2n+1}^2\,(x).$$

Where $L_{k,n+1}$ is the fundamental Lagrange interpolation polynomial, also $L_{k,n+1}$ are given by:

$$L_{k,n+1} = \frac{E_{v,n}^{(\lambda)}}{\acute{E}_{v,n}^{(\lambda)}(\xi_{u,n+1}^{(\lambda)})(x-\xi_{u,n+1}^{(\lambda)})}\,, k = 1,2,\dots,n+1$$

And:

$$L_{v,2n+1} = \frac{F_{v,n}^{(\lambda)}}{\acute{F}_{v,n}^{(\lambda)}(y_{u,n+1}^{(\lambda)})(x-y_{u,n+1}^{(\lambda)})}\,, v = 1,2,\dots,2n+1.$$

For $1 \le p \le \infty$ the usual modulus of continuity of smoothness if definite as:
$\omega_r(f,h,X) = \underbrace{sup}_{0\le t<h} \|\Delta_t^r(f,x)\|_{p,\alpha}$ Also the D.T. modulus of continuity of smoothness if definite as:

$$\omega_{\varphi,p} = \underbrace{sup}_{0<h<t} \left\|\Delta_{h\varphi}f(x)\right\|_{L_{p,\alpha[-1,1]}} \text{ Where:}$$

$$\Delta_{h\varphi}f(x) = f(x + h\varphi(x)/2) + f(x - h\varphi(x)/2).$$

Also, the K-functional is defined by:
$K(f,x)_{p,\alpha} = \inf\{\|f-g\|_{p,\alpha} + t\|\acute{g}\|_{p,\alpha} : g \in W_{p,\alpha}^1\}$ Where $W_{p,\alpha}^1$ is Sobolev space. For the function $f: X \longrightarrow R$ the degree of best one sided approximation of the function $f \in L_{p,\alpha(X)}$ where $X = [-1,1]$ is defined as:
$\tilde{E}_n(f)_{p,\alpha(X)} = \underbrace{inf}_{P_n^{\mp}\in \boldsymbol{P}_n} \|P_n^+ + P_n^-\|_{p,\alpha(X)}$ (Where $\boldsymbol{P}_n$ is the space of all polynomials of degree n and $P_n^-(x) \le f(x) \le P_n^+(x), x \in X$).
And degree of best approximation of f is defined as:

$$E_n(f)_{p,\alpha(X)} = \underbrace{inf}_{P_n\in \boldsymbol{P}_n} \|f - P_n\|_{p,\alpha(X)}\,.$$

AUXILIARY RESULTS

<u>Theorem 2.1:[1]</u>

Let $\lambda \in [0,1], 1 < p < \infty\,, u \in L_p$ and let f be a continuous function on $[-1,1]$ then:

$$\|(H_{n+1}[f](x) - f(x))u(x)\|_p \le \omega_\infty^\varphi(f,\tfrac{1}{n}).$$

If we take limit when $p \to \infty$ get the following theorem:

<u>Theorem 2.2:</u>

Let $\lambda \in [0,1], 1 < p \le \infty\,, u_\alpha(x) \in L_p$ and let $f \in L_{p,\alpha}$-space on $[-1,1]$ then:

$$\|H_{n+1}[f](x) - f(x)\|_{\infty,\alpha} = \|(H_{n+1}[f](x) - f(x))u(x)\|_\infty \le c\omega_{\infty,\alpha}^\varphi(f,\frac{1}{n})$$

<u>Proof:</u>

By using theorem 2.1 and properties of limit we have:

$$\lim_{p\to\infty}\|(H_{n+1}[f](x)-f(x))u(x)\|_p \le \lim_{p\to\infty}\omega_\infty^\varphi(f,\frac{1}{n})$$

$$\|(H_{n+1}[f](x)-f(x))u(x)\|_\infty \le \lim_{p\to\infty}\Big(\underbrace{sup}_{0<h\le n^{-1}}\{\underbrace{sup}_{x\in X}\{\Delta_{\varphi,h}(f,x)\}\}\Big)$$

$$=\underbrace{sup}_{0<h\le n^{-1}}\{\underbrace{sup}_{x\in X}\{\lim_{p\to\infty}\Delta_{\varphi,h}(f,x)\}\}$$

$$\le c\underbrace{sup}_{0<h\le n^{-1}}\{\underbrace{sup}_{x\in X}\{\Delta_{\varphi,h}(f,x)\}\}$$

$$= c\omega_{\infty,\alpha}^\varphi(f,\tfrac{1}{n}).$$

Theorem 2.3:[1]

Let $\lambda\in[0,1], 1<p<\infty, (u\varphi^{2(1-2\lambda)}(x)+1)\in L_p$ and let f be a continuous function on $[-1,1]$ then:

$$\|(H_{2n+1}[f](x)-f(x))u(x)\|_p \le \omega_\infty^\varphi(f,\tfrac{1}{n}).$$

With the same way of proof's theorem 2.2 we get:

Theorem 2.4:

Let $\lambda\in[0,1], 1<p\le\infty, u(x)(\varphi^{2(1-2\lambda)}(x)+1)\in L_p$ and let $f\in L_{p,\alpha}$-space on $[-1,1]$ then:

$$\|(H_{2n+1}[f](x)-f(x))u(x)\|_\infty \le c\omega_{\infty,\alpha}^\varphi(f,\tfrac{1}{n}).$$

As before mention we construct the operators $H_{n+1}^{*,\pm}(f,x_n)$ and $H_{2n+1}^{**,\pm}(f,x_n)$ as:

$$H_{n+1}^{*,\pm}(f,x_n)=H_{n+1}[f](x)\pm\|H_{n+1}[f](x)-f(x)\|_{\infty,\alpha(X)}$$

Also

$$H_{2n+1}^{**,\pm}(f,x_n)=H_{2n+1}[f](x)\pm\|H_{2n+1}[f](x)-f(x)\|_{\infty,\alpha(X)}.$$

Theorem 2.5:

For $f\in L_{p,\alpha}$-space and $\lambda\in[0,1], 1<p\le\infty, u_\alpha(x)\in L_p$ on $[-1,1]$ we have:

$$H_{n+1}^{*,-}(f,x_n)\le f(x)\le H_{n+1}^{*,+}, x\in[-1,1].$$

Proof:

$$\begin{aligned}H_{n+1}^{*,+} &= H_{n+1}[f](x)+\|H_{n+1}[f](x)-f(x)\|_{\infty,\alpha(X)}\\ &= H_{n+1}[f](x)+\|H_{n+1}[f](x)-f(x)\|_{\infty(X)}\\ &\ge H_{n+1}[f](x)+|H_{n+1}[f](x)-f(x)|\\ &= H_{n+1}[f](x)+|f(x)-H_{n+1}[f](x)|\\ &\ge H_{n+1}[f](x)+f(x)-H_{n+1}[f](x)\\ &= f(x).\end{aligned}$$

With the same way we have:

$$H_{n+1}^{*,-}(f,x_n)\le f(x)$$

Also with the same way for the operator $H_{2n+1}^{**,\pm}(f,x_n)$ we have:

$$H_{2n+1}^{**,-}(f,x_n)\le f(x)\le H_{2n+1}^{**,+}, x\in[-1,1].$$

Theorem 2.6: [1]

Let $\lambda\in[0,1]$ and let f be a continuous function on $[-1,1]$ then the uniformly for $x\in[-1,1]$:

$$\underbrace{lim}_{n\to\infty}|H_{n+1}[f](x)-f(x)|=0.$$

Theorem 2.7: [1]

Let $\lambda \in [0,1]$ and let f be a continuous function on $[-1,1]$ then the uniformly for $x \in [-1,1]$:

$$\underbrace{lim}_{n\to\infty} |H_{2n+1}[f](x) - f(x)| = 0.$$

MAIN RESULTS

Theorem 3.1 :(Direct Theorem By using $H^{*,\pm}_{n+1}$-Operator)

For the function $f \in L_{p,\alpha[-1,1]}$-space and $H^{*,\pm}_{n+1}$-operator we have:

$$\tilde{E}_n(f,x)_{p,\alpha} \le c_p \omega^{\varphi}_{\infty,\alpha}(f,\tfrac{1}{n}).$$

Proof:

By the theorems 2.5 and 2.2 we have:

$$\begin{aligned}\tilde{E}_n(f,x)_{p,\alpha} &\le \left\|H^{*,+}_{n+1} - H^{*,-}_{n+1}\right\|_{p,\alpha}\\ &\le \left\|\|H_{n+1}[f](x) - f(x)\|_{\infty,\alpha} + \|H_{n+1}[f](x) - f(x)\|_{\infty,\alpha}\right\|_{p,\alpha}\\ &= 2c_p\|H_{n+1}[f](x) - f(x)\|_{\infty,\alpha}\\ &\le 2c_p\omega^{\varphi}_{\infty,\alpha}\left(f,\frac{1}{n}\right) = c_p\omega^{\varphi}_{\infty,\alpha}(f,\tfrac{1}{n}).\end{aligned}$$

Theorem 3.2 :(Direct Theorem By using $H^{**,\pm}_{2n+1}$-Operator)

For the function $f \in L_{p,\alpha[-1,1]}$-space and $H^{**,\pm}_{2n+1}$-operator we have:

$$\tilde{E}_n(f,x)_{p,\alpha} \le c_p \omega^{\varphi}_{\infty,\alpha}(f,\tfrac{1}{n}).$$

Proof:

By the theorems 2.5 and 2.4 we have:

$$\begin{aligned}\tilde{E}_n(f,x)_{p,\alpha} &\le \left\|H^{**,+}_{2n+1} - H^{**,-}_{2n+1}\right\|_{p,\alpha}\\ &\le \left\|\|H_{2n+1}[f](x) - f(x)\|_{\infty,\alpha} + \|H_{2n+1}[f](x) - f(x)\|_{\infty,\alpha}\right\|_{p,\alpha}\\ &= 2c_p\|H_{2n+1}[f](x) - f(x)\|_{\infty,\alpha}\\ &\le 2c_p\omega^{\varphi}_{\infty,\alpha}\left(f,\frac{1}{n}\right) = c_p\omega^{\varphi}_{\infty,\alpha}(f,\tfrac{1}{n}).\end{aligned}$$

Now, we discuss the uniform convergence of the $H^{*,\pm}_{n+1}$and $H^{**,\pm}_{2n+1}$-operators as the following:

Theorem 3.3:

Let $\lambda \in [0,1], 1 < p < \infty\,, u \in L_p$ and let $f \in L_{p,\alpha[-1,1]}$ then the uniformly convergence for $x \in [-1,1]$ is:

$$\underbrace{lim}_{n\to\infty} \left|H^{*,+}_{n+1} - H^{*,-}_{n+1}\right| = 0.$$

Proof:

As in proof of theorem 3.1, properties of limit and theorem 2.6, we get:

$$\begin{aligned}\underbrace{lim}_{n\to\infty} \left|H^{*,+}_{n+1} - H^{*,-}_{n+1}\right| &= \underbrace{lim}_{n\to\infty} \left|2(\|H_{n+1}[f](x) - f(x))\|_{\infty,\alpha}\right|\\ &= 2\underbrace{lim}_{n\to\infty} \|H_{n+1}[f](x) - f(x))\|_{\infty,\alpha}\\ &= 2\underbrace{lim}_{n\to\infty}\underbrace{sup}_{x\in X}\{|H_{n+1}[f](x) - f(x)|\}\\ &= 2\underbrace{\sup}_{x\in X}\{\underbrace{lim}_{n\to\infty} |H_{n+1}[f](x) - f(x)|\}\\ &= 2\underbrace{\sup}_{x\in X}\{(0) = 2(0) = 0\}.\end{aligned}$$

With the same way we can prove the following theorem:

Theorem 3.4:

Let $\lambda \in \left[0,\frac{1}{2}\right]$, $1 < p < \infty$, $u \in L_p$ and let $f \in L_{p,\alpha[-1,1]}$ then the uniformly convergence for $x \in [-1,1]$ is:

$\underbrace{lim}_{n\to\infty} \left|H_{2n+1}^{**,+} - H_{2n+1}^{**,-}\right| = 0.$

Theorem 3.5: [2]

For $f \in L_{p[-1,1]}$, $1 \le p \le \infty$ we have:

$E_n(f)_\infty \le \tilde{E}_n(f)_\infty \le 2E_n(f)_\infty$.

With the same way of prove above theorem we can prove the following theorem:

$E_n(f)_p \le \tilde{E}_n(f)_p \le 2E_n(f)_p$.

Theorem 3.6:

For $f \in L_{p[-1,1]}$, $1 \le p \le \infty$ we have:

$E_n(f)_{p,\alpha} \le \tilde{E}_n(f)_{p,\alpha} \le 2E_n(f)_{p,\alpha}$.

Proof:

Since fu is bounded function and by using above theorem we get:

$E_n(f)_{p,\alpha} = E_n(fu)_p \le \tilde{E}_n(fu)_p = \tilde{E}_n(f)_{p,\alpha}$

And:

$\tilde{E}_n(f)_{p,\alpha} = \tilde{E}_n(fu)_p \le 2E_n(fu)_p = 2E_n(f)_{p,\alpha}$.

The following theorem is proved in **[4]** for Hermite-Fejer polynomials by using zeros of first kind Chebyshev polynomial $T_n(x) = \cos(ncos^{-1}x)$, $-1 \le x \le 1$

And:

$x_k = cos\theta_k, \theta_k = \frac{2k-1}{2n}, k = 1,2, \dots, n$.

Now, we shall prove same theorem by using zeroes of Stieltjes polynomial $E_{n+1}^{(\lambda)}$:

Theorem 3.7:

For $f \in L_{p,\alpha[-1,1]}$, $1 \le p \le \infty$ we have:

$$\tau(f,\Delta(\cdot,\tfrac{1}{n}))_{p,\alpha} \le \frac{c}{n}\sum_{s=0}^{n} \begin{cases} \|f - H_s\|_{p,\alpha} + \left\|\check{f} - H_s\right\|_{p,\alpha} & if\ p = 1,\infty \\ \|f - H_s\|_{p,\alpha} & if\ 1 < p < \infty \end{cases}.$$

Before, we prove the above theorem we needed to prove the following lemmas:

Lemma 3.8: [4]

Let $f, g \in L_{p,\alpha}$, $(1 \le p \le \infty)$, $\delta > 0$ then:

$\tau_k(f,\delta)_{p,\alpha} \le \tau_k((f-g),\delta)_{p,\alpha} + \tau_k(g,\delta)_{p,\alpha}$.

Lemma 3.9: [4]

Let $f, g \in L_{p,\alpha}$, $(1 \le p \le \infty)$, $\delta > 0$ then:

$\tau_k(f,\delta)_{p,\alpha} \le c\|f\|_{p,\alpha}$.

Lemma 3.10: [4]

If $f, \acute{f} \in L_{p,\alpha}$, $(1 \le p \le \infty)$, $\delta > 0$ then:

$\tau_k(f,\delta)_{p,\alpha} \le \delta\left\|\acute{f}\right\|_{p,\alpha}$.

Lemma 3.11: [4]

Let $f \in L_{p,\alpha(X)}$ such that $(1\le p < \infty)$ then:

$$K(f,\tfrac{1}{n},l_{p,\alpha},\omega_{p,\alpha}^{1}) \le \tfrac{c}{n}\sum_{s=0}^{n}\begin{cases}E_s(f)_{p,\alpha}+E_s(\check{f})_{p,\alpha} & if\ p=1,\infty\\ E_s(f)_{p,\alpha} & if\ 1<p<\infty\end{cases}.$$

<u>Lemma 3.12:</u> [4]

If $f \in L_{p,\alpha},(1 \le p \le \infty), and\ 0 \le\ \acute{\delta} < \delta$ then:
$\tau_k(f,\acute{\delta})_{p,\alpha} \le \tau_k(f,\delta)_{p,\alpha}$.

<u>Proof of theorem 3.7:</u>

By using lemmas (3.8), (3.9) and (3.10), we get:

$$\tau(f,\frac{1}{n})_{p,\alpha} \le \tau((f-g),\frac{1}{n})_{p,\alpha} + \tau(g,\frac{1}{n})_{p,\alpha}$$
$$\le c\|f-g\|_{p,\alpha} + \tfrac{1}{n}\|\acute{g}\|_{p,\alpha}$$
$$=K(f,\tfrac{1}{n},L_{p,\alpha},\omega_{p,\alpha}^{1})$$

Now, by using lemma (3.11) we get:

$$\tau(f,\tfrac{1}{n})_{p,\alpha} \le \tfrac{c}{n}\sum_{s=0}^{n}\begin{cases}E_s(f)_{p,\alpha}+E_s(\tilde{f})_{p,\alpha} & if\ p=1,\infty\\ E_s(f)_{p,\alpha} & if\ 1<p<\infty\end{cases}$$
$$\le \tfrac{c}{n}\sum_{s=0}^{n}\begin{cases}\|f-H_s\|_{p,\alpha}+\left\|\check{f}-H_s\right\|_{p,\alpha} & if\ p=1,\infty\\ \|f-H_s\|_{p,\alpha} & if\ 1<p<\infty\end{cases}$$

By using lemma (3.12) we have:

$$\tau(f,\Delta(\cdot,\frac{1}{n}))_{p,\alpha} \le \tau(f,\frac{1}{n})_{p,\alpha}$$

And the proof is complete.

<u>Theorem 3.13:</u>

For $f \in L_{p,\alpha[-1,1]}, 1 \le p \le \infty$ and $H_{n+1}^{*,\pm}$-operator we have:

$$\tau(f,\tfrac{1}{n})_{p,\alpha} \le \tfrac{c}{n}\sum_{s=0}^{n}\begin{cases}\tilde{E}_s(f)_{p,\alpha}+\tilde{E}_s(\tilde{f})_{p,\alpha} & if\ p=1,\infty\\ \tilde{E}_s(f)_{p,\alpha} & if\ 1<p<\infty\end{cases}.$$

<u>Proof:</u>

By using theorems 3.6 and 3.7 we get:

$$\tau(f,\tfrac{1}{n})_{p,\alpha} \le \tfrac{c}{n}\sum_{s=0}^{n}\begin{cases}\tilde{E}_s(f)_{p,\alpha}+\tilde{E}_s(\tilde{f})_{p,\alpha} & if\ p=1,\infty\\ \tilde{E}_s(f)_{p,\alpha} & if\ 1<p<\infty\end{cases}.$$

Now, we try to estimate a degree of best one-sided approximation of the derivative of the function $f \in L_{p,\alpha[-1,1]}, 1 \le p \le \infty$.

The following theorem is proved in **[3]** for the function $f \in C[a,b]$ by using trigonometric polynomials in this part we shall prove same theorem for $f \in L_{p,\alpha[-1,1]}$ by using algebraic polynomial.

<u>Lemma 3.14:</u> [3]

For each real p three is a constant M_p with the property that for each sequence $0 < u_k \le u_{k+1} \le \cdots \le u_l$, such that $2 \le u_i/u_{i-1} \le 4$ for $k < i \le l$, and for each positive decreasing function $\phi(u)$ defined for $u \ge 0$(or at least for all values u_i and all $u = 0,1,\dots$),

$\sum_{i=k}^{l} u_i^p \phi(u_i) \leq M_p \sum_{[\frac{1}{2}u_k] \leq n < u_l} (n+1)^{p-1} \phi(n).$

Theorem 3.15:

For $f \in L_{p,\alpha[-1,1]}$, $1 < p \leq \infty$ and $r \geq 1$, $\sum_{n=1}^{\infty} n^{r-1} E_n(f) < +\infty$

Then $f^{(r)}$ exists and is continuous, and its degree of approximation satisfies

$$E_n(f^r) \leq c_r \sum_{[n/2]}^{\infty} k^{r-1} E_k(f);$$

The constant M_r depends only on r.

Proof:

If $P_k(x)$ is the polynomial of best approximation of degree k for $f(x)$ then

$$f(x) - P_n(x) = \sum_{i=1}^{\infty} \{P_{2^i n}(x) - P_{2^{i-1}n}(x)\}$$

This series converges uniformly. The series obtained by formal differentiation

$$f^{(s)} = P_n^{(s)}(x) + \sum_{i=1}^{\infty} \{P_{2^i n}^{(s)}(x) - P_{2^{i-1}n}^{(s)}(x)\}, \qquad 1 \leq s \leq r$$

Also, converge uniformly as a consequence, since the norm of the i th term does not exceed (i.e. if $\left\|P_n^{(r)}\right\|_p \leq Mn^r$).

$2^{is} n^s \left\|P_{2^i n}(x) - P_{2^{i-1}n}(x)\right\|_{p,\alpha} \leq 2.2^{is}.n^s \phi(2^{i-1}n)$,Where $\phi(n) = E_n(f)$.

And the convergence of the series $\sum (2^{i-1}n)^r \phi(2^{i-1}n)$ follows from lemma 3.14, by now a theorem about uniformly convergent series $f^{(r)}(x)$ exists and is equal to the sum of the last series this proving the first part of the theorem.

For the second part, we can now write

$$\begin{aligned}
\left\|f^{(r)} - P_n^{(r)}\right\|_{p,\alpha} &= \left\|\left(f^{(r)} - P_n^{(r)}\right)u(x)\right\|_p \leq \sum_{i=1}^{\infty} \left\|(P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})u(x)\right\|_p \\
&= \sum_{i=1}^{\infty} \left\|P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)}\right\|_{p,\alpha} \\
&= \sum_{i=1}^{\infty} \left\|(P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})u_\alpha(x)\right\|_p \\
&= \sum_{i=1}^{\infty} \left\|\boldsymbol{P}_{2^i n}^{(r)} - \boldsymbol{P}_{2^{i-1}n}^{(r)}\right\|_p \\
&\leq c \sum_{i-1}^{\infty} (2^{i-1}n)^r \phi(2^{i-1}n) \\
&\leq c \sum_{[n/2]}^{\infty} k^{r-1} \phi(k).
\end{aligned}$$

BY lemma 3.14, the proof is completes.

Theorem 3.16: [5]

For $f \in L_p$-space, $0 < p \leq \infty$ and positive integer k, there exists an algebraic polynomial P_n of degree $\leq n$ such that:

$E_n(f)_p \leq \|f - P_n\|_p \leq c_k \omega_k^{\varphi}(f, n^{-1})_p.$

Theorem 3.17:

For $f \in L_{p,\alpha}$-space, $0 < p \leq \infty$ and positive integer k, there exists an algebraic polynomial P_n of degree $\leq n$ such that:

$E_n(f)_{p,\alpha} \leq \|f - P_n\|_{p,\alpha} \leq c_k \omega_k^{\varphi}(f, n^{-1})_{p,\alpha}.$

Proof:

By using theorems 3.16 we get:

$$E_n(f)_{p,\alpha} \leq \|f-P_n\|_{p,\alpha} = \|(f-P_n)\omega_\alpha\|_p \leq c_k\omega_k^{\varphi}(f\omega_\alpha, n^{-1})_p$$
$$\leq c_k\omega_k^{\varphi}(f, n^{-1})_{p,\alpha}.$$

Theorem 3.18:

For $f \in L_{p,\alpha[-1,1]}$, $1 \leq p \leq \infty$ and $H_{n+1}^{*,\pm}$-operator we have:

$$\tilde{E}_n(\acute{f}, x)_{p,\alpha} \leq (2n-1)\omega_{\infty,\alpha}^{\varphi}(f, \tfrac{1}{n}).$$

Proof:

By using theorem 2.5, Bernstein enquality and theorem 3.1 and since:

$H_{n+1}^{*,-}(f, x_n) \leq f(x) \leq H_{n+1}^{*,+}$, so we have:

Either $\acute{H}_{n+1}^{*,-}(f, x_n) \leq \acute{f}(x) \leq \acute{H}_{n+1}^{*,+}$ or $\acute{H}_{n+1}^{*,+}(f, x_n) \leq \acute{f}(x) \leq \acute{H}_{n+1}^{*,-}$

So we have:

$$\tilde{E}_n(\acute{f}, x)_{p,\alpha} \leq \left\|\acute{H}_{n+1}^{*,+}(f,x) - \acute{H}_{n+1}^{*,-}(f,x)\right\|_{p,\alpha}$$
$$\leq \left\|(H_{n+1}^{*,+}(f,x) - H_{n+1}^{*,-}(f,x))\acute{}\right\|_{p,\alpha}$$
$$\leq (2n-1)\left\|H_{n+1}^{*,+}(f,x) - H_{n+1}^{*,-}(f,x)\right\|_{p,\alpha}$$
$$\leq (2n-1)\omega_{\infty,\alpha}^{\varphi}(f, \tfrac{1}{n}).$$

The other cases and by using theorems 3.15, 3.17 we have:

$$\tilde{E}_n(\acute{f}, x)_{p,\alpha} = E_n(\acute{f}, x)_{p,\alpha} \leq \left\|\acute{f}(x) - \acute{H}_{n+1}^{*,+}(f,x))\right\|_{p,\alpha}$$
$$= \left\|\acute{f}(x) - \acute{H}_{n+1}[f](x) - (\|f(x) - H_{n+1}[f](x)\|_{\infty,\alpha})\acute{}\right\|_{p,\alpha}$$
$$= \left\|\acute{f}(x) - \acute{H}_{n+1}[f](x)\right\|_{p,\alpha}$$
$$\leq c_k \sum_{[n/2]}^{\infty} k^{r-1}\omega_k^{\varphi}(f, n^{-1})_{p,\alpha}.$$

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