European Journal of Statistics and Probability

Vol.4, No.3, pp.12-19, July 2016

Published by European Centre for Research Training and Development UK (www.eajournals.org)

DECOMPOSITIONS OF SUM-SYMMETRY MODEL FOR ORDINAL SQUARE CONTINGENCY TABLES

Kouji Yamamoto[1], Mana Aizawa[2] and Sadao Tomizawa[2]

[1]Department of Clinical Epidemiology and Biostatistics, Graduate School of Medicine, Osaka University

[2]Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science

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ABSTRACT: *For ordinal square contingency tables, Yamamoto, Tanaka and Tomizawa (2013) proposed the sum-symmetry model and gave a decomposition of it. The present paper proposes three kinds of extended sum-symmetry models, and gives two another decompositions of the sum-symmetry model using these models. Audiometry data are analyzed.*

KEYWORDS: Decomposition, Model, Square Contingency Table, Sum-Symmetry

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INTRODUCTION

Consider an $r \times r$ square contingency table with the same ordinal row and column categories. Let p_{ij} denote the probability that an observation will fall in the i-th row and j-th column of the table ($i = 1, \ldots, r$; $j = 1, \ldots, r$). For analysis of contingency tables, Bowker (1948) considered the symmetry (S) model defined by

$$p_{ij} = p_{ji} \text{ for } i = 1, \ldots, r; j = 1, \ldots, r.$$

McCullagh (1978) proposed the conditional symmetry (CS) model defined by

$$p_{ij} = \delta p_{ji} \text{ for } i < j,.$$

where the parameter δ is unspecified. A special case of the CS model obtained by putting $\delta = 1$ is the S model.

Yamamoto, Tanaka and Tomizawa (2013) considered the sum-symmetry (SS) model defined by

$$\sum_{(i,j)\in R(t)} \sum p_{ij} = \sum_{(i,j)\in R(t)} \sum p_{ji} \text{ for } t = 3, \ldots, 2r-1,$$

where

$$R(t) = \{(i,j) | i + j = t, i < j\}.$$

Let X denote the row variable and Y denote the column variable of the table. The SS model is also expressed as

$$\Pr(X + Y = t, X < Y) = \Pr(X + Y = t, X > Y) \text{ for } t = 3, \ldots, 2r-1.$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

When $r = 3$, the SS model is equivalent to the S model.

Yamamoto et al. (2013) also proposed the conditional SS (CSS) model defined by

$$\sum_{(i,j)\in R(t)} \sum p_{ij} = \Delta \sum_{(i,j)\in R(t)} \sum p_{ji} \text{ for } t = 3, \dots, 2r-1,$$

where the parameter Δ is unspecified. This model is also expressed as

$$\Pr(X+Y=t, X<Y) = \Delta \Pr(X+Y=t, X>Y) \text{ for } t = 3, \dots, 2r-1.$$

A special case of the CSS model obtained by putting $\Delta = 1$ is the SS model. When $r = 3$, the CSS model is equivalent to the CS model.

The global symmetry (GS) model (Read, 1977) is defined by

$$\sum_{i<j} \sum p_{ij} = \sum_{i<j} \sum p_{ji}.$$

This model is also expressed as

$$\Pr(X<Y) = \Pr(X>Y).$$

Yamamoto et al. (2013) gave the decomposition of the SS model as follows :

Theorem 1. *The SS model holds if and only if both the CSS and GS models hold.*

Consider the audiometry data in Tables 1 and 2. Table 1 is the data of hearing threshold at 8000 Hz in decibels of 1872 examinees aged 20-69 in National Health and Nutrition Examination Survey (2001-2002). Table 2 is the data of hearing threshold at 3000 Hz in decibels of 1670 examinees aged 20-69 in National Health and Nutrition Examination Survey (1999-2000). In Tables 1 and 2 the row variable is the right ear threshold grade and column value is the left ear threshold grade. The category in both Tables 1 and 2 means thresholds of hearing impairment severity based on National Center for Health Statistics, Audiometry/Tympanometry Procedures Manual (2009): (1) is ≤ 25 dB (i.e., Normal hearing), (2) is 26-40 dB (i.e., Mild hearing loss), (3) is 41-55 dB (i.e., Moderate hearing loss), and (4) is 56+ dB (i.e., Moderately severe or greater hearing loss).

It seems natural to see the degree of an individual's hearing threshold grade as the sum of the grades of both right and left ears, and therefore it would be meaningful to apply the SS and CSS models for these data.

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Table 1. Hearing threshold at 8000 Hz in decibels of 1872 examinees aged 20-69, National Health and Nutrition Examination Survey (2001-2002). (Upper and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the ESS and 2PSS models, respectively.)

Right ear threshold grade	Left ear threshold grade				Total
	(1)	(2)	(3)	(4)	
(1)	1120 (1120.00) (1120.00)	120 (124.27) (123.10)	22 (20.62) (20.52)	15 (14.06) (14.04)	1277
(2)	115 (110.73) (111.90)	80 (80.00) (80.00)	55 (51.54) (51.49)	15 (15.95) (15.99)	265
(3)	15 (16.38) (16.48)	33 (36.46) (36.51)	46 (46.00) (46.00)	49 (50.58) (50.86)	143
(4)	9 (9.94) (9.96)	11 (10.05) (10.01)	30 (28.42) (28.14)	137 (137.00) (137.00)	187
Total	1259	244	153	216	1872

(1) ≤ 25 dB, (2) 26-40 dB, (3) 41-55 dB, and (4) 56+ dB

Table 2. Hearing threshold at 3000 Hz in decibels of 1670 examinees aged 20-69, National Health and Nutrition Examination Survey (1999-2000). (The parenthesized values are the maximum likelihood estimates of expected frequencies under the CD model.)

Right ear threshold grade	Left ear threshold grade				Total
	(1)	(2)	(3)	(4)	
(1)	1249 (1249.00)	73 (73.69)	31 (30.91)	15 (14.89)	1368
(2)	65 (64.05)	59 (59.00)	23 (22.83)	8 (7.90)	155
(3)	9 (9.02)	13 (13.18)	32 (32.00)	13 (12.78)	67
(4)	6 (6.09)	9 (9.20)	18 (18.46)	47 (47.00)	80
Total	1329	154	104	83	1670

(1) ≤ 25 dB, (2) 26-40 dB, (3) 41-55 dB, and (4) 56+ dB

In this paper, we propose three kinds of extended SS models and give two kinds of decompositions of the SS model using these models.

New models

Consider an extension of the SS model as follows:

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\sum_{(i,j)\in R(t)}\sum p_{ij} = \Theta^{t-2} \sum_{(i,j)\in R(t)}\sum p_{ji} \text{ for } t = 3, \dots, 2r-1,$$

where the parameter Θ is unspecified. We shall refer to this model as the exponential SS (ESS) model. This model may be expressed as

$$\Pr(X+Y=t, X<Y) = \Theta^{t-2}\Pr(X+Y=t, X>Y) \text{ for } t = 3, \dots, 2r-1.$$

A special case of the ESS model obtained by putting $\Theta = 1$ is the SS model.

For the audiometry data in Tables 1 and 2, X is the right ear threshold grade and Y is the left ear threshold grade. For the audiometry data in Tables 1 and 2, the ESS model states that the probability that the degree of the hearing threshold grade for an individual with the right ear threshold grade being less than his/her left ear threshold grade, is t (t = 3, ... ,7), is Θ^{t-2} times higher than the probability that the degree of it with the right ear threshold grade being greater than his/her left ear threshold grade, is t.

Next, consider an extension of the ESS model as follows:

$$\sum_{(i,j)\in R(t)}\sum p_{ij} = \Delta\Theta^{t-2} \sum_{(i,j)\in R(t)}\sum p_{ji} \text{ for } t = 3, \dots, 2r-1,$$

where the parameters Δ and Θ are unspecified. We shall refer to this model as the 2-parameters SS (2PSS) model. This model is also expressed as

$$\Pr(X+Y=t, X<Y) = \Delta\Theta^{t-2}\Pr(X+Y=t, X>Y) \text{ for } t = 3, \dots, 2r-1.$$

Special cases of the 2PSS model obtained by putting $\Delta = 1$ and by putting $\Theta = 1$ are the ESS and CSS models, respectively. Also, a special case of the 2PSS model obtained by putting $\Delta = \Theta = 1$ is the SS model.

For the audiometry data, the 2PSS model states that the probability that the degree of the hearing threshold grade for an individual with the right ear threshold grade being less than his/her left ear threshold grade, is t (t = 3, ... ,7), is $\Delta\Theta^{t-2}$ times higher than the probability that the degree of it with the right ear threshold grade being greater than his/her left ear threshold grade, is t.

Moreover, consider the model defined by

$$\Pi_c = \Pi_d,$$

where

$$\Pi_c = 2\sum_{3\le k<l\le 2r-1}\sum p_{1(k)}p_{2(l)},$$

$$\Pi_d = 2\sum_{3\le k<l\le 2r-1}\sum p_{2(k)}p_{1(l)},$$

$$p_{1(t)} = \sum_{(i,j)\in R(t)}\sum p_{ij} = \Pr(X+Y=t, X<Y),$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$p_{2(t)} = \sum_{(i,j)\in R(t)}\sum p_{ji} = \Pr(X+Y=t, X>Y).$$

For a randomly selected pair of observations, (i) Π_c indicates that the probability that the members that ranks higher (i.e., takes *l* rather than *k* (< *l*)) on *X*+*Y* ranks higher (i.e., takes positive rather than negative) on *X*-*Y*, and (ii) Π_d indicates that the probability that the members that ranks higher on *X*+*Y* ranks lower on *X*-*Y*. We shall refer to this model as the concordance-discordance (CD) model.

Decompositions of the SS Model

We obtain a new decomposition of the SS model as follows:

Theorem 2. *The SS model holds if and only if both the ESS and GS models hold.*

Proof. If the SS model holds, then ESS and GS models hold. Assuming that both the ESS and GS models hold, then we shall show that the SS model holds. From the assumption that the ESS model holds, we have

$$\sum_{t=3}^{2r-1}\sum_{(i,j)\in R(t)}\sum p_{ij} = \sum_{t=3}^{2r-1}\Theta^{t-2}\sum_{(i,j)\in R(t)}\sum p_{ji}. \qquad (1)$$

Since the GS model holds, we have

$$\sum_{t=3}^{2r-1}\sum_{(i,j)\in R(t)}\sum p_{ij} = \sum_{t=3}^{2r-1}\sum_{(i,j)\in R(t)}\sum p_{ji}. \qquad (2)$$

From equations (1) and (2), we obtain $\Theta = 1$. Namely the SS model holds. The proof is completed.

We also obtain the following theorem:

Theorem 3. *The SS model holds if and only if all the 2PSS, CD and GS models hold.*

Proof. If the SS model holds, then the 2PSS, CD and GS models hold. Assuming that the 2PSS, CD and GS models hold, then we shall show that the SS model holds. From the assumption that the 2PSS model holds, we have

$$p_{1(t)} = \Delta\Theta^{t-2}p_{2(t)} \text{ for } t = 3, \ldots, 2r-1. \qquad (3)$$

Thus we see

$$\Pi_c = 2\sum_{3\le k<l\le 2r-1}\sum \Delta\Theta^{k-2}p_{2(k)}p_{2(l)},$$

and

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$\Pi_d = 2 \sum_{3 \le k < l \le 2r-1} \sum \Delta \Theta^{l-2} p_{2(k)} p_{2(l)}.$$

Since the CD model holds, we have

$$\sum_{3 \le k < l \le 2r-1} \sum \Theta^{k-2} p_{2(k)} p_{2(l)} = \sum_{3 \le k < l \le 2r-1} \sum \Theta^{l-2} p_{2(k)} p_{2(l)}. \qquad (4)$$

The equation (4) holds if and only if $\Theta = 1$. Therefore, by equation (3), we obtain

$$p_{1(t)} = \Delta p_{2(t)} \text{ for } t = 3, \dots, 2r-1. \qquad (5)$$

In addition, since the GS model holds, we have

$$\sum_{t=3}^{2r-1} p_{1(t)} = \sum_{t=3}^{2r-1} p_{2(t)}.$$

Using the equation (5), we obtain $\Delta = 1$, thus $p_{1(t)} = p_{2(t)}$ for t = 3, ..., 2r - 1. Namely the SS model holds. The proof is completed.

Goodness-Of-Fit Test

Let n_{ij} denote the observed frequency in the i-th row and j-th column of the table ($i = 1, \dots, r; j = 1, \dots, r$). Assume that a multinomial distribution is applied to the $r \times r$ table. The maximum likelihood estimates (MLEs) of expected frequencies under the ESS, 2PSS and CD models could be obtained by using the Newton-Raphson method in the log-likelihood equation. Each model can be tested for goodness-of-fit by, e.g., the likelihood ratio chi-squared statistic with corresponding degrees of freedom (df). The likelihood ratio statistic for testing goodness-of-fit of model M is given by

$$G^2(M) = 2 \sum_{i=1}^{r} \sum_{j=1}^{r} n_{ij} \log\left(\frac{n_{ij}}{\hat{m}_{ij}}\right),$$

where $\hat{m}_{ij}$ is the MLE of expected frequency m_{ij} under model M. The numbers of df for the SS, ESS, 2PSS, and CD models are $2r-3$, $2(r-2)$, $2r-5$, and 1, respectively.

Examples

Example 1

Consider the data in Table 1. From Table 3 we see that the SS, GS and CD models fit the data in Table 1 poorly. However, the CSS, ESS and 2PSS models fit these data well. Therefore, it is seen from Theorem 2 that the poor fit of the SS model is caused by the influence of the lack of structure of the GS model rather than the ESS model. Also, it is seen from Theorem 3 that the poor fit of the SS model is caused by the influence of the lack of structures of the GS and CD models rather than the 2PSS model.

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Table 3. Likelihood ratio statistic G^2 values for models applied to the data in Tables 1 and 2.

Applied models	For Table 1			For Table 2		
	df	G^2	*p*-value	df	G^2	*p*-value
SS	5	13.746*	0.017	5	20.588*	0.001
ESS	4	1.522	0.823	4	17.126*	0.002
CSS	4	5.607	0.230	4	14.029*	0.007
2PSS	3	1.493	0.684	3	13.706*	0.003
CD	1	4.692*	0.030	1	0.048	0.827
GS	1	8.139*	0.004	1	6.559*	0.010

* means significant at the 0.05 level

Since the CSS and 2PSS models fit these data well, we shall test the hypothesis that the CSS model holds assuming that the 2PSS model holds. According to the test based on the difference between the G^2 values for the CSS and 2PSS models (i.e., $G^2(CSS) - G^2(2PSS) = 4.114$ with 1 df), this hypothesis is rejected at the 0.05 significance level. Therefore the 2PSS model is preferable to the CSS model.

Similarly, we shall test the hypothesis that the ESS model holds assuming that the 2PSS model holds. According to the test based on the difference between the G^2 values for the ESS and 2PSS models (i.e., $G^2(ESS) - G^2(2PSS) = 0.029$ with 1 df), the ESS model is preferable to the 2PSS model.

Under the ESS model, the MLE of Θ is 1.122. Thus, under the ESS model, the probability that the degree of the hearing threshold grade for an individual whose left ear threshold grade is greater than his/her right ear threshold, is t ($t = 3, \dots, 7$), is estimated to be $(1.122)^{t-2}$ times higher than the probability that the degree of it whose right ear threshold grade is greater than his/her left ear threshold grade, is t.

Example 2

Consider the data in Table 2. From Table 3 we see that the SS model fits the data in Table 2 poorly. Also the CD model fits these data very well but the other models fit these data poorly. We can see from Theorem 3 that the poor fit of the SS model is caused by the influence of the lack of structures of the GS and 2PSS models rather than the CD model.

Under the CD model, for a randomly selected pair of individuals, (i) the probability that the members that ranks higher (i.e., takes l rather than k ($< l$)) on the degree of the hearing level (as the sum of the grades of both right and left ears) ranks higher (i.e., takes the right ear threshold grade being greater than the left ear threshold grade) on the right ear threshold grade minus the left ear threshold grade, is equal to (ii) the probability that the members that ranks higher on the degree of the hearing level ranks lower (i.e., takes the right ear threshold grade being less than the left ear threshold grade) on the right ear threshold grade minus the left ear threshold grade.

Concluding remarks

In this paper, we have proposed decompositions for the SS model, into the ESS and GS models (i.e., Theorem 2) and into the 2PSS, CD and GS models (i.e., Theorem 3).

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Theorems 1, 2 and 3 may be useful for seeing a reason for the poor fit of the SS model when the SS model fits the data poorly. Especially, Theorem 3 rather than Theorem 1 would be useful for seeing in more details the reason for poor fit of the SS model when the SS model fits the data poorly.

Acknowledgements

The authors are greatly thankful to two reviewers and the editor for their helpful and valuable comments.

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