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CHARACTERIZATION OF IDEAL SEMIGROUPS OF INVERSE SEMIGROUPS

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ABSTRACT: *In this paper mainly we have obtained characterization for ideal Semigroups of inverse semigroups.*

KEYWORDS: Ideal Semigroup, Fundamental Semigroup, Fundamental Inverse Semigroup, Munnsemigroup , Ideal Congruence.

AMS Subject classification No: 06B10,20M 17 , 20M 18

INTRODUCTION

In 1991 Garcia J.I. In his paper entitled "The congruence extension property for algebraic semigroups" which is called an ideal congruence .In his paper he defined an ideal semigroup as a semigroup in which every congruence is an ideal congruence(Rees congruence) and has studied about the properties of ideal semigroups.In this paper we study the ideal semigroups on inverse semigroups. In this connection first it is observed that if an inverse semigroup S is an ideal semigroup then it is fundamental. Further it is shown that a necessary and sufficient condition for an inverse semigroup S to be an ideal semigroup is that S is fundamental and every non-zero element of the semilattice E(S) of S is maximal. It is interesting to note that if an orthodox semigroup S is an ideal semigroup then it is an inverse semigroup.A regular semigroup S is said to satisfy the condition(A) if for any idempotents $e,f,g,h \in S$, $e(fg)h=0$ if and only if $e(fg)^2h=0$. We provide an example to show that every regular semigroup with zero need not satisfy this condition.It is easy to observe that the condition (A) is productive and hereditary. In this paper it is observed that if a regular semigroup S with condition (A) is an ideal semigroup then it is an inverse semigroup. If a semilattice E an ideal semigroup then the necessary and sufficient condition for every subsemigroup of the Munnsemigroup T_E to be an ideal semigroup is obtained.Finally it is proved that if S is an inverse semigroup then a necessary and sufficient condition for every sub semigroup of S to be an ideal semigroup is that S is fundamental , every non-zero element of the semilattice E of S is maximal and S has atmost two idempotent elements.

First we start with the following preliminaries.

Def[1]:- Acongruence relation ρ on a semigroup Sis said to be an ideal congruence if there exists an ideal I of S such that $\rho = (I \times I) \cup 1_s$.

Def[2]:- A semigroup S is said to be an ideal semigroup if every congruence on S is an ideal congruence .

Def[3]:- A regular semigroup S is said to be an inverse semigroup if every element of S has unique inverse (or) A regular semigroup S is said to be an inverse semigroup if the set of all idempotents of S is a semilattice.

Def[4]:- An inverse semigroup S is said to be fundamental if the maximum idempotent separating congruence $\mu=\{(a,b)\in S \times S \,/\, a'ea= b'eb, \text{ for all } e\in E(S)\}$ is 1_s.

Def[5]:- An inverse sub semigroup T of an inverse semigroup S is said to be full inverse sub semigroup if it contains all its idempotents.

First we start with the following lemma.

Lemma1:-If an inverse semigroup S is an ideal semigroup then it is fundamental.

Proof:- Let S be an inverse semigroup. Suppose S is an ideal semigroup. Consider the idempotent separating congruence $\mu=\{(a,b)\in S \times S \,/\, a'ea= b'eb, \text{ for all } e\in E(S)\}$.Since S is an ideal semigroup,we have $\mu = (I \times I) \cup 1_s$, where $I=\mu(0)$ is an ideal of S.Let $a \in I=\mu(0)$ so that $(a,0) \in \mu$ and hence by definition $a'ea=0e0$ for all $e\in E(S)$.In particular for $e= aa'$ we have $a'aa'a=0=0\ aa'0=0$ and hence $a'a=0$ so that $a= aa'a=a.0=0$.There fore $I=\{0\}$ and hence $\mu = 1_s$.Thus S is fundamental.

Let E be a semilattice and let $e\in E$, then $Ee=\{i\in E:i\leq e\}$ is a partial ideal of E. The uniformity relation $\upsilon =\{(e,f)\in E \times E:Ee \cong Ef\}$. For each $(e,f)\in \upsilon$, $T_{e,f}$ is defined to be the set of all isomorphisms from Ee onto Ef and $T_E=\{ T_{e,f} : (e,f)\in \upsilon\}$.

Lemma2:-If E is a semilattice of the form

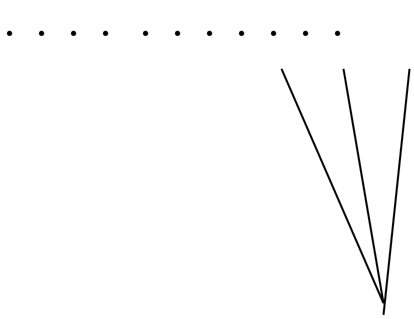

Fig: 1

Then $|T_{e,f}|=1$ for all e,f $\in$ E-{0},

Proof:-Let E be a semilattice of the form Fig:1. Let e,f $\in$ E-{0},since Ee={0,e} and Ef={0,f} there is only one isomorphism between Ee and Ef which $0 \mapsto 0$ and $e \mapsto f$ hence $|T_{e,f}|=1$.

Lemma3:-If S is a fundamental semigroup then the semilattice E of idempotents of S is also fundamental.

Proof:-Let S be a fundamental inverse semigroup and E be the semilattice of idempotents of S. Since S is fundamental we have $\mu = 1_s$.Let (e,f) $\in \mu_E$ then e.g.e=f.g.f, for all g$\in$ E. In particular for g=e and g=f we have e.e=f.e and g=e ,g=f. We have e.e=f.e and e.f= f.f, therefore e=f.e=e.f=f as E is a semilattice. Thus $\mu_E = 1_E$ and hence E is fundamental.

Theorem4:- If S is an inverse semigroup with semilattice E of idempotents of S is of the form (1) then μ= H

Proof:- We have $\mu \subseteq H$ by proposition 5.3.7 in j.m.howie. Let (a,b) $\in$ H and a$\neq$ b then both a and b are non zero then aa′= bb′ and a′a= b′b.The mapping α_a:E aa′$\rightarrow$ E a′a defined by $x\alpha_a$= a′ea is an isomorphism .Similarly α_b:E bb′$\rightarrow$ E b′b is an isomorphism. Since E is of the form fig(1), by using lemma 1.2, for any e,f$\in$ E-{0} there is only one isomorphism from Ee to Ef. Since a$\neq$b, we have aa′= bb′, a′a= b′b $\in$ E-{0}. Hence $\alpha_a = \alpha_b$,.now for e$\in$E, we have a′ea= a′ea a′a= a′(ea a′)a= a′(ebb′)a= (eb b′)α_a=(eb b′)α_b= b′ eb b′b= b′eb.There fore a′ea= b′eb for all e$\in$ E and hence (a,b) $\in \mu$ thus μ= H .

Theorem5:-An inverse semigroup S is fundamental and the semilattice E is of the form fig (1) if and only if inverse semigroup S and every fundamental inverse semigroup of S are ideal semigroups.

Proof:-Let S be an inverse semigroup. Suppose that S is fundamental and the semilattice E is of the form fig (1).Then for any e,f$\in$ E $T_{e,f}$ =φ or $|T_{e,f}|$ =1.Since E is of the form fig (1) , by using Theorem 4, we have μ= H and since S is fundamental $\mu=1_s$ and hence H==1_s.Let ρ be a congruence on S and let (a,b) $\in \rho$.If a a′$\neq$b b′then since aa′,bb′$\in$ E and E is of the form fig (1), we have a a′=0=b b′.We have (a,b) $\in \rho$ imply that (aa′a, aa′b) $\in \rho$ and hence (a, aa′b) $\in \rho$.Now aa′b= aa′b b′b=0.There fore(a,0) $\in \rho$ so that a$\in$ ρ(0).Similarly if a′a$\neq$ b′b we can observe that a,b$\in$ ρ(0). On the other hand if a a′=b b′ and a′a= b′b then(a,b) $\in$H==1_s.There fore for any (a,b) $\in \rho$ we have either a,b$\in$ ρ(0) or (a,b) $\in 1_s$.There fore ρ=(ρ(0) X ρ(0)) $\cup$ 1_s and hence ρ is a congruence on S.Thus S is an ideal semigroup.Now let T be any fundamental inverse semigroup of S. Then E(T) is of the form fig(1) and hence T is an ideal semigroup.

Conversely suppose that S is an ideal semigroup and every fundamental inverse subsemigroup of S is an ideal semigroup. Then S is fundamental and hence by lemma3 E is a fundamental inverse sub semigroup of S. Therefore by assumption E is an ideal semigroup and hence E is of the form fig(1).

In the following theorem it is shown that if S is an orthodox and ideal semigroup then it is an inverse semigroup.

Theorem6:-If an orthodox semigroup S is an ideal semigroup then S is an inverse semigroup.

Proof:- Let S be an orthodox semigroup. Consider the smallest inverse semigroup congruence υ = {(x,y) $\in$S X S/ V(x)= V(y)} in S. Since S is an ideal semigroup we have υ=(I X I) $\cup 1_s$ and I=υ(0).For any a$\in$I=υ(0), we have (a,0) $\in\upsilon$ and hence υ(a)= υ(0)={0}.The only inverse of a is 0 so that a=a.0.a=0.Therefore I={0} and υ= 1_s Thus 1_s is an inverse congruence on S.Therefore S/1_s is an inverse semigroup and S $\cong$S/1_s.Hence S is an inverse semigroup.

Denition6:-Let S be a regular semigroup with zero.Then S is said to satisfy the condition(A) if for any e,f,g,h $\in$ E(S), e(fg)h=0$\Leftrightarrow$ e$(fg)^2$h=0.

Theorem7:- Let S be a regular semigroup with zero satisfy the condition (A).If S is an ideal semigroup then S is an inverse semigroup.

Proof:- Let S be a regular semigroup with zero satisfy the condition (A), suppose that S is an ideal semigroup. Let σ be the congruence generated by the relation R={(ef), $(ef)^2$/e,f$\in$ E(S)}.It is easy to observe that σ is the smallest orthodox congruence on S, obviously σ is an orthodox congruence on S.Let ρ be any orthodox congruence on S and let e,f$\in$ E(S),as ρ is an orthodox congruence we have (ef), $(ef)^2$$\in\rho$ and hence R $\subseteq\rho$.Since σ is the congruence generated by R we have $\sigma \subseteq \rho$ therefore σ is the smallest orthodox congruence on S. Now we observe that S is an inverse semigroup .since S is an ideal semigroup

$\sigma=(\sigma(0) X\sigma(0)) \cup 1_s$. Let $a\in \sigma(0)$ then $(a,0) \in \sigma$ and $\sigma=(R^c)^e$ by j.m.howie. If $a\neq 0$, then there exists a sequence $a=a_1\rightarrow a_2\rightarrow\ldots\rightarrow a_{n-1}\rightarrow a_n=0$ of elementary R-transitions connecting a to 0.For each i, either $(a_i,a_{i+1}) \in R^c$ or$(a_{i+1}, a_i) \in R^c$ where i=1 to n-1.For i=n-1, either $(a_{n-1},0) \in R^c$ or$(0, a_{n-1}) \in R^c$. Suppose that $(a_{n-1},0) \in R^c$ then $a_{n-1}=p(ef)q$ and $0=p(ef)^2q$, for some $p,q\in S^1= S\cup\{1\}$ and $e,f\in E(S)$.Let $p'\in V(P)$, $q'\in V(q)$, we have $0= p'.0.\ q'= p'p(ef)^2q\ q'$.since $p'p$, $q\ q'$,e,f E(S),by our assumption we have $p'p(ef)\ q\ q'=0$.Now $a^{n-1}= p(ef)\ q=p\ p'p(ef)\ q\ q'q=p(p'p\ (ef)\ q\ q')q=p.o.q=0$.Onthe other hand , if$(0,a_{n-1}) \in R^c$, then $0= p(ef)\ q$ and $a_{n-1}= p(ef)^2q$ for some $p,q\in S^1$ and $e,f \in E(S)$.).Let $p'\in V(P)$, $q'\in V(q)$ we have $0=p'.0.\ q'= p'p(ef)\ q\ q'$ and $p'p$, $q\ q'$, e,f E(S), there fore by assumption $pp'p(ef)^2q\ q'q=0$ imply that $a_{n-1}= p(ef)^2q= pp'p(ef)^2q\ q'q=0$. In any case we have that $a_{n-1}=0$. Now repeating the argument we get $a_{n-2}=0$, continueing we have a=0.There fore $\sigma=1_s$ and σ is an orthodox congruence and hence $S/1_s$ is orthodox. There fore S is orthodox, thus by using thorem 6 , S is an inverse semigroup. The following example shows that any regular semigroup with zero need not satisfy the condition (A).

Example 8: Let S={e,f,h,I,o} be a regular semigroup with zero with multiplication **.** is defined as follows:

.	e	f	h	i	o
e	e	f	o	o	o
f	e	f	f		o
h	i	h	h	i	o
i	i	h	o	o	o
o	o	o	o	o	O

Here h,e E(S), such that $he=i\neq 0$ and $(he)^2=i^2=0$ so that $h(he)e=h^2e^2=he=i$ and $h(he)^2e=h0e=0$. Therefore S does not satisfy the condition A.

It is natural ask whether a regular semigroup with zero satisfy the condition A is orthodox?

The following is an example to show that it need not be true because of the following example.

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Example9:Considerthe semigroup $S=\{(-1)^n 2^n a, (-1)^n 2^n b, (-1)^n 2^n c, (-1)^n 2^n d / n \in z+ \cup \{0\}\} \cup \{(-1/2)^n a, (-1/2)^n b, (-1/2)^n c, (-1/2)^n d / n \in z+\} \cup \{(-1)^n 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} / n \in z+ \cup \{0\}\} \cup \{(-1/2)^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} / n \in z+\}$ where $a=\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$, $b=\begin{bmatrix} 1 & 32 \\ 0 & 0 \end{bmatrix}$ c=ab and d=ba with respect to the usual matrix multiplication. Clearly a and b are idempotents. For any non negative integer n, the inverses of $(-1)^n 2^n a$ and $(-1)^n 2^n b$ are $(-1/2)^n a$ and $(-1/2)^n b$ respectively. Here $c^2= - 2c$ and c.1/4c.c=c similarly $d^2= -2d$ and d. 1/4d.d =d. Clearly every element of S is regular and hence S is regular semigroup. Also a,b E(S) such that a.b and b.a are not idempotents and hence S is not an orthodox semigroup, since S has no zero element adjoin zero element of S, then S^o is a regular semigroup with zero satisfy the condition (A) but not orthodox.

Lemma10:-If S is a regular semigroup with zero satisfy the condition (A) then every subsemigroup of S with zero also satisfies the condition (A).

Proof:- Let S be a regular semigroup with zero satisfy the condition (A) and let T be a subsemigroup of S.Let $e,f,g,h \in E(T) \subseteq E$ and S satisfies the condition (A).Therefore $e(fg)h=0 \Leftrightarrow e(fg)^2h=0$ and hence T also satisfies the condition(A).

Lemma11:-If $\{S_\alpha\}_{\alpha\in\Delta}$ is a family of regular semigroups with zero then each $S\alpha$ satisfies the condition (A) if and only if the direct product of $\{S_\alpha\}_{\alpha\in\Delta}$ satisfies the condition (A).

Remark12:- If $\{S_\alpha\}_{\alpha\in\Delta}$ is a family of regular semigroups then $S=\{f\in\prod S_\alpha / f(\alpha)=0$, for almost all $\alpha\in\Delta$, except for finite α in $\Delta\}$ is also a regular subsemigroup.

In the following theorem it is obtained that if E is a semilattice of the form fig(1) then every inverse subsemigroup of T_E is also an ideal semigroup.

Theorem13:-If E is a semilattice of the form form fig(1), then every inverse subsemigroup of T_E is also an ideal semigroup.

Proof:- Let E be a semilattice of the form form fig(1) , then E is an ideal semigroup by Theorem 1.5, By theorem 5.4.1 in J.M.HOWIE, T_E is an inverse semigroup with semilattice of idempotents isomorphic to E.Also by corollary 5.4.6 in J.M.HOWIE T_E is fundamental , by theorem 1.5 T_E, and every fundamental inverse subsemigroup of T_E are also ideal semigroups.It is observed that every inverse subsemigroup of T_E is fundamental. Hence every inverse sub semigroup of T_E is also an ideal semigroup.

It is interestimg to note that every full inverse subsemigroup of a fundamental inverse subsemigroup of a fundamental inverse subsemigroup is also a fundamental.

Theorem14:-Every full inverse subsemigroup of a fundamental inverse subsemigroup is fundamental.

Proof:- Let S be a fundamental inverse semigroup, then $\mu_s = 1_s$.Let T be a full inverse subsemigroup of S.Then E(S) $\subseteq$ T so that E(T) = E(S) .Let $(a,b)\in\mu_T$ then $aea' = beb'$ for all $e\in$ E(T) = E(S) and hence $(a,b)\in\mu_S$ and $\mu_s = 1_s$ thus $\mu_T = 1_T$ and hence T is fundamental.

Corollary 15:-Every full inverse subsemigroup of a fundamental inverse subsemigroup whose semi lattice of idempotents E is of the form fig(1) is an ideal semigroup.

Proof:- Let S be a fundamental inverse semigroup such that semi lattice E of idempotents of S is of the form fig(1) then by theorem 15 S is an ideal semigroup. Let T be a full inverse subsemigroup of S, then by the above Theorem T is fundamental and hence theorem 14 T is also an ideal semigroup.

Lemma16 :- If E is a semilattice of the form fig(1) then for any $\alpha\in T_E$ either $a^2=\alpha$ or $\alpha=0$.

Theorem17:- If E is a semilattice of the form fig(1), then every subsemigroup of T_E is ideal semigroup iff | E|<4.

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