CALIBRATING AUXILIARY DIFFERENTIAL EQUATION TO SOLVE THE BENJAMIN-BONA-MOHONY EQUATION

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ABSTRACT: *In this paper we propose a new method called calibrating auxiliary differential equation to establish exact solutions for the Benjamin-Bona-Mohony equation. Among the obtained exact solutions are solitary and periodic wave solutions of nonlinear evolution equations. The proposed calibrating auxiliary differential equation method is straight forward and powerful mathematical tool that could be used for solving other nonlinear partial differential equations.*

KEYWORDS: Benjamin-Bona-Mohony equation, Auxiliary Differential Equation, Traveling Wave Solutions, Exact solutions.

INTRODUCTION

Nonlinear evolution equations (NLEEs) play an important role in science because it is used to model various complex phenomena. Thus, obtaining and analyzing the exact solutions of these equations contributes to describe and understand the dynamic aspect of the phenomena under consideration. Among the well-known NLEEs which attracted many researchers is the Benjamin-Bona-Mohony (BBM) equation.Many different approaches have been proposed to find the exact solutions of the NLEEs equations and in particular to BBM equation. Among these methods that have been widely used are: F-expansion method [6], $\left(\frac{G'}{G}\right)$ expansion method [7], the sine-cosine function method [8-9], the tanh-function method [10], the exp-function method [11], the Jacobi elliptic function method [12], auxiliary equation method [13-15].

In this paper we propose a new efficient method that is used to find the exact solutions of the BBM equation. The method, that we call calibrating auxiliary differential equation, is based on the auxiliary differential equation. The remaining of the paper is organized as follows. In Section 2 the new method is described, followed in Section 3 by its applications to obtain exact solutions of the BBM equation. Section 4 shows the graphics of some obtained exact solutions. Finally, Section 5 concludes the paper.

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The Calibrating Auxiliary Differential Equation method

In the following, we introduce the main steps of the calibrating auxiliary equation method

Step 1: Suppose that a nonlinear partial differential equation is given by

$$F(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xtt}, u_{tx}, u_{ttx} \ldots \ldots) = 0, \quad (2.1)$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in u and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Using the following generalized wave transformation:

$$u(x,y,t) = u(\xi), \xi = k_1 x + k_2 t + \xi_0, \quad (2.2)$$

where k_1, k_2 and ξ_0 are a constant, Then Eq. (2.1) is reduced to the following ODE:

$$P\left(u, k_1 u', k_2 u', k_1^2 u'', k_2^2 u'', \ldots \ldots \ldots\right) = 0, \quad (2.3)$$

where $\left(' = \frac{d}{d\xi}\right)$and P is a polynomial in u and its total derivatives.

Step2. We suppose that Eq. (2.3) has the following formal solution:

$$u(\xi) = \sum_{i=1}^{N} \alpha_i \left(F(\xi)\right)^i \quad (2.4)$$

where N is a positive integer, $\alpha_i (i = 1,2, \ldots)$ are constants, and the function $F(\xi)$ satisfies a nonlinear ordinary differential equation:

$$\frac{d}{d\xi} F(\xi) = \left(A_0 + A_1 F(\xi)\right)\sqrt{B_0 + B_1 F(\xi) + B_2 F^2(\xi)} \quad (2.5)$$

Step3. Determine the positive integer N in (2.4) by balancing the highest order derivatives and nonlinear terms in Eq. (2.3).

Step4. Substituting (3.4) along with (2.5) into (2.3) and equating the coefficients of $\sqrt{B_0 + B_1 F(\xi) + B_2 F^2(\xi)}\left(F(\xi)\right)^j (j = 0,1,2..)$ to be zero, yields a set of algebraic equations for $A_0, A_1, B_0, B_1, B_2, k_1, k_2, \xi_0$ and $\alpha_i (i = 0,1,2,..N)$.

Step5. Solving these algebraic equations by Maple or Mathematica, we get the values of $A_0, A_1, B_0, B_1, B_2, k_1, k_2, \xi_0$ and $\alpha_i (i = 0,1,2,..N)$.

Step6. Substituting the values A_0, A_1, B_0, B_1, B_2 into (2.5) , and then solving the resulting a nonlinear ordinary differential equation we can obtain the $F(\xi)$.

Step7. Substituting the $F(\xi), k_1, k_2, \xi_0$ and $\alpha_i (i = 0,1,2,..N)$ into (2.4) we can obtain the exact solutions of Eq. (2.3).

3 Exact solutions of the BBM Equation

In this section, we apply proposed method to study well known Benjamin-Bona-Mohony equation. Let us consider the following BBM equation:

$$u_t + \delta_0 u_x + \delta_1 u u_x - \delta_2 u_{xxt} = 0 \quad (3.1)$$

By balancing (uu_x) and (u_{xxt}) in Eq (3.1) we obtain $(n = 2)$.

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Therefore, the solution of Eq (3.1) is of the form :

$$u(\xi) = u(x,t) = \alpha_0 + \alpha_1 F(\xi) + \alpha_2 F^2(\xi), \quad \xi = k_1 x + k_2 t + \xi_0, \tag{3.2}$$

where $F(\xi)$ satisfies Eq (2.5), α_0, α_1, and α_2 are constants to be determined later. Substituting Eq(3.2) together with Eq(2.5), the left - hand side is converted into polynomials in $\sqrt{B_0 + B_1 F(\xi) + B_2 F^2(\xi)}\left(F(\xi)\right)^j (j = 0,1,2,3,4)$. We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations for A_0, A_1, B_0, B_1, B_2, k_1, k_2, ξ_0, α_0, α_1 and α_2. Solving these algebraic equations using algebraic software Maple, we obtain following.

Case 1:

$$A_0 = 0,\ A_1 = A_1,\ B_2 = 0,\ k_1 = k_1,\ k_2 = k_2,\ \alpha_0 = \alpha_0,\ \alpha_1 = \alpha_1,\ \alpha_2 = 0,\ \xi_0 = \xi_0,$$
$$B_0 = \left(\frac{k_2 + \delta_0 k_1 + \alpha_0 \delta_1 k_1}{\delta_2 k_2 k_1^2 A_1^2}\right),\ B_1 = \left(\frac{\alpha_1 \delta_1}{3\delta_2 k_1 k_2 A_1^2}\right) \tag{3.3}$$

Substituting (3.3) in to (2.5), we have

$$\frac{d}{d\xi} F(\xi) = A_1 F(\xi) \sqrt{B_0 + B_1 F(\xi)} \tag{3.4}$$

Using Maple, some solutions of Eq (3.4) are:

1) if $B_1 = 0$,

$$F_{1,1}(\xi) = c.\exp\left(A_1 \sqrt{B_0}\xi\right) \tag{3.5}$$

2) if $B_0 = 0,\ A_1 \neq 0,\ B_1 \neq 0.$

$$F_{1,2}(\xi) = \frac{4}{A_1^2 B_1 (\xi + c)^2} \tag{3.6}$$

3) if $B_0 \prec 0,\ B_1 \neq 0.$

$$F_{1,3}(\xi) = -\left(\frac{B_0}{B_1}\right)\left[1 + \tan^2\left(\frac{A_1}{2}\sqrt{-B_0}(\xi + c)\right)\right] \tag{3.7}$$

4) if $B_0 \succ 0,\ B_1 \neq 0.$

$$F_{1,4}(\xi) = -4\left(\frac{B_0}{B_1}\right)\frac{\exp\left(A_1\sqrt{B_0}(\xi + c)\right)}{\left(\exp\left(A_1\sqrt{B_0}(\xi + c)\right) + 1\right)^2} \tag{3.8}$$

Substituting (3.3) and (3.5) into (3.2) we find

$$u_{1,1}(\xi) = \alpha_0 + \alpha_1 c\exp\left(A_1 \sqrt{\left(\frac{k_2 + \delta_0 k_1 + \alpha_0 \delta_1 k_1}{\delta_2 k_2 k_1^2 A_1^2}\right)}\xi\right) \tag{3.9}$$

where $\xi = k_1 x + k_2 t + \xi_0$,

In particular setting

$$k_1 = 1, k_2 = 1,\ \alpha_0 = 1, \alpha_1 = 2, \xi_0 = 0, \delta_0 = 1, \delta_1 = 0, \delta_2 = 2, c = 1, A_1 = 1$$

we find

$$u_{1,1}^0(\xi) = 1 + 2\exp(\xi); \qquad \xi = x + t \qquad (See\ Figure\ 1). \tag{3.10}$$

Substituting (3.3) and (3.6) into (3.2) we find

$$u_{1,2}(\xi) = \alpha_0 + \alpha_1 \frac{4}{A_1^2\left(\frac{\alpha_1 \delta_1}{3\delta_2 k_1 k_2 A_1^2}\right)(\xi + c)^2} \tag{3.11}$$

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where $\xi = k_1x + k_2t + \xi_0$,
In particular setting

$$k_1 = 1, k_2 = -2, \alpha_0 = 1, \alpha_1 = 6, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_1 = 1$$

We find

$$u_{1,2}^0(\xi) = 1 - \frac{24}{\xi^2}; \qquad \xi = x - 2t \qquad (See\ Figure\ 2). \tag{3.12}$$

Substituting (3.3) and (3.7) into (3.2) we find

$$u_{1,3}(\xi) = \alpha_0 - 3\alpha_1\left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1}{\alpha_1\delta_1k_1}\right)\left[1 + \tan^2\left(\frac{A_1}{2}\sqrt{-\left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1}{\delta_2k_2k_1^2A_1^2}\right)}(\xi + c)\right)\right] \tag{3.13}$$

where $\xi = k_1x + k_2t + \xi_0$,
In particular setting

$$k_1 = 1, k_2 = -3, \alpha_0 = 1, \alpha_1 = 3, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = -\frac{1}{3}, c = 0, A_1 = 1$$

We find

$$u_{1,3}^0(\xi) = 4 + 3\tan^2\left(-\frac{\xi}{2}\right); \qquad \xi = x - 3t \qquad (See\ Figure\ 3). \tag{3.14}$$

Substituting (3.3) and (3.8) into (3.2) we find

$$u_{1,4}(\xi) = \alpha_0 - 12\alpha_1\left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1}{\alpha_1\delta_1k_1}\right)\frac{\exp\left(A_1\sqrt{\left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1}{\delta_2k_2k_1^2A_1^2}\right)}(\xi+c)\right)}{\left(\exp\left(A_1\sqrt{\left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1}{\delta_2k_2k_1^2A_1^2}\right)}(\xi+c)\right)+1\right)^2} \tag{3.15}$$

where $\xi = k_1x + k_2t + \xi_0$,
In particular setting

$$k_1 = 1, k_2 = 1, \alpha_0 = 1, \alpha_1 = 9, \xi_0 = 1, \delta_0 = 1, \delta_1 = 1, \delta_2 = 3, c = -1, A_1 = 1$$

We find

$$u_{1,4}^0(\xi) = 1 - \frac{36\exp(\xi-1)}{(\exp(\xi-1)+1)^2}; \qquad \xi = x + t + 2^{\pi} \qquad (See\ Figure\ 4) \tag{3.16}$$

Case 2:

$$A_0 = A_0, A_1 = A_1, B_1 = B_1, B_2 = 0, k_1 = k_1, k_2 = k_2, \alpha_0 = \alpha_0, \alpha_2 = 0, \xi_0 = \xi_0 \tag{3.17}$$

$$B_0 = \left(\frac{k_2+\delta_0k_1+\alpha_0\delta_1k_1-2\delta_2k_2k_1^2A_0A_1B_1}{\delta_2k_2k_1^2A_1^2}\right), \qquad \alpha_1 = \left(\frac{3\delta_2k_1k_2A_1^2B_1}{\delta_1}\right)$$

Substituting (3.12) in to (2-5), we have

$$\frac{d}{d\xi}F(\xi) = \left(A_0 + A_1F(\xi)\right)\sqrt{B_0 + B_1F(\xi)} \tag{3.18}$$

Using Maple, the solutions of Eq (3.18) are:

1) if $A_0 = 0, B_0 = 0, B_1 \neq 0, A_1 \neq 0.$

$$F_{2,1}(\xi) = \frac{4}{B_1A_1^2(\xi+c)^2} \tag{3.19}$$

2) if $A_0 = 0, B_0 \prec 0, B_1 \neq 0,$

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$$F_{2,2}(\xi) = -\left(\frac{B_0}{B_1}\right)\left[1 + \tan^2\left(\frac{A_1\sqrt{-B_0}}{2}(\xi + c)\right)\right] \tag{3.20}$$

3) $A_0 = 0, B_0 \succ 0, B_1 \neq 0,$

$$F_{2,3}(\xi) = -4\left(\frac{B_0}{B_1}\right)\frac{\exp(A_1\sqrt{B_0})}{\left(\exp(A_1\sqrt{B_0})+1\right)^2} \tag{3.21}$$

4) if $B_0 = 0, A_1 \neq 0, \frac{A_0 B_1}{A_1} \succ 0,$

$$F_{2,4}(\xi) = \frac{A_0}{A_1}\tan^2\left[\frac{A_1}{2}\sqrt{\frac{A_0 B_1}{A_1}}(\xi + c)\right] \tag{3.22}$$

5) if $B_0 = 0, A_1 \neq 0, \frac{A_0 B_1}{A_1} \prec 0,$

$$F_{2,5}(\xi) = -\left(\frac{A_0}{A_1}\right)\left[\frac{\exp\left(A_1\sqrt{-\frac{A_0 B_1}{A_1}}(\xi+c)\right)+1}{\exp\left(A_1\sqrt{-\frac{A_0 B_1}{A_1}}(\xi+c)\right)-1}\right]^2 \tag{3.23}$$

6) if $A_0 = B_0, A_1 = B_1, A_1 \neq 0,$

$$F_{2,6}(\xi) = -\frac{A_0 A_1^2(\xi+c)^2 - 4}{A_1^3(\xi+c)^2} \tag{3.24}$$

7) if $(A_0 B_1 - A_1 B_0) = 0, A_1 \neq 0, B_1 \neq 0$

$$F_{2,7}(\xi) = -\frac{A_1^2 B_0(\xi+c)^2 - 4}{B_1 A_1^2(\xi+c)^2} \tag{3.25}$$

8) if $\left(\frac{A_0 B_1 - A_1 B_0}{A_1}\right) = \Delta_1 \succ 0, A_1 \neq 0, B_1 \neq 0,$

$$F_{2,8}(\xi) = -\left(\frac{B_0}{B_1}\right) + \left(\frac{\Delta_1}{B_1}\right)\tan^2\left(\frac{A_1}{2}\sqrt{\Delta_1}(\xi + c)\right) \tag{3.26}$$

9) if $\left(\frac{A_0 B_1 - A_1 B_0}{A_1}\right) = -\Delta_1 \prec 0, A_1 \neq 0, B_1 \neq 0,$

$$F_{2,9}(\xi) = -\frac{\left(\frac{A_0 B_1}{A_1}\right)\left[\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)-1\right]^2 + 4B_0\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)}{B_1\left[\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)+1\right]^2} \tag{3.27}$$

Substituting (3.17) and (3.19) into (3.2) we find

$$u_{2,1}(\xi) = \alpha_0 + \left(\frac{12\delta_2 k_1 k_2 A_1^2 B_1}{\delta_1 B_1 A_1^2}\right)\frac{1}{(\xi+c)^2} \tag{3.28}$$

where $\xi = k_1 x + k_2 t + \xi_0.$
In particular setting
$k_1 = 1, k_2 = 2, \alpha_0 = -1, \xi_0 = 0, \delta_0 = -1, \delta_1 = \delta_2 = 1, c = 0, A_0 = 0, A_1 = 3, B_1 = 1$

we find

$$u_{2,1}^0(\xi) = -1 + \frac{24}{\xi^2}; \qquad \xi = x + 2t \qquad (See\ Figure\ 5) \tag{3.29}$$

Substituting (3.17) and (3.20) into (3.2) we find

$$u_{2,2}(\xi) = \alpha_0 - \left(\frac{3\delta_2 k_1 k_2 A_1^2 B_0}{\delta_1}\right)\left[1 + \tan^2\left(\frac{A_1\sqrt{-B_0}}{2}(\xi + c)\right)\right] \tag{3.30}$$

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where $\xi = k_1x + k_2t + \xi_0$,
In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = -3, \xi_0 = 0, \delta_0 = 1, \delta_1 = \delta_2 = 1, c = 0, A_0 = 0, A_1 = 1, B_1 = -1$

We find

$$u_{2,2}^0(\xi) = 3\tan^2\left(\frac{1}{2}\xi\right); \qquad \xi = x + t \qquad (See\ Figure\ 6) \tag{3.31}$$

Substituting (3.17) and (3.21) into (3.2) we find

$$u_{2,3}(\xi) = \alpha_0 - \left(\frac{12\delta_2 k_1 k_2 A_1^2 B_0}{\delta_1}\right)\frac{\exp\left(A_1\sqrt{B_0}(\xi+c)\right)}{\left(\exp\left(A_1\sqrt{B_0}(\xi+c)\right)+1\right)^2} \tag{3.32}$$

Where

$$\xi = k_1x + k_2t + \xi_0,$$

In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = 1, \xi_0 = 1, \delta_0 = 1, \delta_1 = 1, \delta_2 = 3, c = -1, A_0 = 0, A_1 = 1, B_1 = 2$
We find

$$u_{2,3}^0(\xi) = 1 - 36\frac{\exp(\xi-1)}{(\exp(\xi-1)+1)^2}; \qquad \xi = x + t + 1 \qquad (See\ \ Figure\ 7) \tag{3.33}$$

Substituting (3.17) and (3.22) into (3.2) we find

$$u_{2,4}(\xi) = \alpha_0 + \left(\frac{3\delta_2 k_1 k_2 A_0 A_1 B_1}{\delta_1}\right)\tan^2\left[\frac{A_1}{2}\sqrt{\frac{A_0 B_1}{A_1}}(\xi + c)\right] \tag{3.34}$$

where$\xi = k_1x + k_2t + \xi_0$,
In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = -1, \xi_0 = \pi, \delta_0 = 1, \delta_1 = 1, \delta_2 = \frac{1}{2}, c = 0, A_0 = 1, A_1 = 1, B_1 = 1$
We find

$$u_{2,4}^0(\xi) = -1 + \frac{3}{2}\cot^2\left(\frac{1}{2}\xi\right); \qquad \xi = x + t + \pi \qquad (See\ Figure\ 8) \tag{3.35}$$

Substituting (3.17) and (3.23) into (3.2) we find

$$u_{2,5}(\xi) = \alpha_0 - \left(\frac{3\delta_2 k_1 k_2 A_0 A_1 B_1}{\delta_1}\right)\left[\frac{\exp\left(A_1\sqrt{-\frac{A_0 B_1}{A_1}}(\xi+c)\right)+1}{\exp\left(A_1\sqrt{-\frac{A_0 B_1}{A_1}}(\xi+c)\right)-1}\right]^2$$

where $\xi = k_1x + k_2t + \xi_0$,
In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = 2, \xi_0 = 0, \delta_0 = \delta_1 = -1, \delta_2 = 1, c = 0, A_0 = 1, A_1 = -1, B_1 = 1$

we find

$$u_{2,5}^0(\xi) = 2 - 3\left(\frac{\exp(-\xi)+1}{\exp(-\xi)-1}\right)^2; \qquad \xi = x + t \qquad (See\ Figure\ 9) \tag{3.36}$$

Substituting (3.17) and (3.24) into (3.2) we find

$$u_{2,6}(\xi) = \alpha_0 - \left(\frac{3\delta_2 k_1 k_2 B_1}{\delta_1 A_1}\right)\left(\frac{A_0 A_1^2(\xi+c)^2 - 4}{(\xi+c)^2}\right) \tag{3.37}$$

where $\xi = k_1x + k_2t + \xi_0$,
In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = 1, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = \frac{1}{18}, c = 1, A_0 = 2, A_1 = 3, B_1 = 3$

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We find

$$u_{2,6}^{0}(\xi) = 1 - \frac{18(\xi+1)^2-4}{6(\xi+1)^2}; \qquad \xi = x + t \qquad (See\ Figure\ 10) \tag{3.38}$$

Substituting (3.17) and (3.25) into (3.2) we find

$$u_{2,7}(\xi) = \alpha_0 - \left(\frac{3\delta_2 k_1 k_2}{\delta_1}\right)\left(\frac{A_1^2 B_0(\xi+c)^2-4}{(\xi+c)^2}\right) \tag{3.39}$$

Where $\xi = k_1 x + k_2 t + \xi_0$,
In particular setting
$k_1 = 1, k_2 = 1, \alpha_0 = 4, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_0 = 2, A_1 = 1, B_1 = 1$

We find

$$u_{2,7}^{0}(\xi) = 4 - 6\frac{(\xi^2-2)}{\xi}; \qquad \xi = x + t \qquad (See\ Figure\ 11) \tag{3.40}$$

Substituting (3.17) and (3.26) into (3.2) we find

$$u_{2,8}(\xi) = \alpha_0 + \left(\frac{3\delta_2 k_1 k_2 B_0 A_1^2}{\delta_1}\right) \times \left[\left(-\frac{B_0}{B_1}\right) + \left(\frac{A_0 B_1 - A_1 B_0}{B_1 A_1}\right)\tan^2\left(\frac{A_1}{2}\sqrt{\left(\frac{A_0 B_1 - A_1 B_0}{A_1}\right)}(\xi + c)\right)\right] \tag{3.41}$$

(46)
Where $\xi = k_1 x + k_2 t + \xi_0$,
In particular setting

$$k_1 = 1, k_2 = 1, \alpha_0 = 3, \xi_0 = \frac{\pi}{2}, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_0 = 1, A_1 = 1, B_1 = 2$$

We find

$$u_{2,8}^{0}(\xi) = 3\tan^2\left(\frac{\xi}{2}\right); \qquad \xi = x + t + \frac{\pi}{2} \qquad (See\ Figure\ 12) \tag{3.42}$$

Substituting (3.17) and (3.27) into (3.2) we find

$$u_{2,9}(\xi) = \alpha_0 + \left(\frac{3\delta_2 k_1 k_2 B_1 A_1^2}{\delta_1}\right)\left[-\frac{\left(\frac{A_0 B_1}{A_1}\right)\left[\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)-1\right]^2 + 4B_0\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)}{B_1\left[\exp\left(A_1\sqrt{\Delta_1}(\xi+c)\right)+1\right]^2}\right] \tag{3.43}$$

Where $\xi = k_1 x + k_2 t + \xi_0 \quad ; \quad \Delta_1 = \left(\frac{A_0 B_1 - A_1 B_0}{A_1}\right)$
In particular setting

$$k_1 = 1, k_2 = 1, \alpha_0 = 2, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_0 = 1, A_1 = 1, B_1 = 1$$

We find

$$u_{2,9}^{0}(\xi) = 2 - 3\frac{[(\exp(\xi)-1)^2 + 8\exp(\xi)]}{[\exp(\xi)+1]^2}; \qquad \xi = x + t \qquad (See\ Figure\ 13) \tag{3.44}$$

Case3 :

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$$A_0 = 0, A_1 = A_1, B_2 = B_2, B_1 = 0, k_1 = k_1, k_2 = k_2, \alpha_0 = \alpha_0, \alpha_1 = 0, \xi_0 = \xi_0$$
$$B_0 = \left(\frac{k_2 + \delta_0 k_1 + \alpha_0 \delta_1 k_1}{4\delta_2 k_2 k_1^2 A_1^2}\right), \qquad \alpha_2 = \left(\frac{12\delta_2 k_1 k_2 A_1^2 B_2}{\delta_1}\right) \tag{3.45}$$

Substituting (3.45) in to (2.5), we have

$$\frac{d}{d\xi}F(\xi) = A_1 F(\xi)\sqrt{B_0 + B_2 F^2(\xi)} \tag{3.46}$$

Using Maple, some solutions of Eq (3.46) are:
1) if $A_1 \neq 0, B_2 \neq 0, B_0 = 0,$

$$F_{3,1}(\xi) = \frac{1}{\left(A_1\sqrt{B_2}(\xi + c)\right)} \tag{3.47}$$

2) if $B_0 \succ 0, B_2 \succ 0,$

$$F_{3,2}(\xi) = \frac{2\exp\left(-A_1\sqrt{B_0}(\xi + c)\right)}{-\left(\frac{B_2}{B_0}\right) + \exp\left(-2A_1\sqrt{B_0}(\xi + c)\right)} \tag{3.48}$$

3) if $B_0 \prec 0, B_2 \succ 0, A_1 \neq 0,$

$$F_{3,3}(\xi) = \frac{-1}{\tan\left(A_1\sqrt{-B_0}(\xi + c)\right)\sqrt{\frac{-B_2}{B_0\left(\tan^2\left(A_1\sqrt{-B_0}(\xi + c)\right) + 1\right)}}} \tag{3.49}$$

Substituting (3.45) and (3.47) into (3.2) we find

$$u_{3,1}(\xi) = \alpha_0 + \left(\frac{12\delta_2 k_1 k_2 A_1^2 B_2}{\delta_1}\right)\left[\frac{1}{\left(A_1\sqrt{B_2}(\xi + c)\right)}\right]^2 \tag{3.50}$$

Where $\xi = k_1 x + k_2 t + \xi_0,$
In particular setting
$$k_1 = 1, k_2 = 1, \alpha_0 = 1, \xi_0 = 3, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_1 = 1, B_2 = 1$$
We find

$$u_{3,1}^0(\xi) = 1 - \frac{6}{\xi^2}; \qquad \xi = x + t + 3 \qquad (See\ Figure\ 14) \tag{3.51}$$

Substituting (3.45) and (3.48) into (3.2) we find

$$u_{3,2}(\xi) = \alpha_0 + \left(\frac{12\delta_2 k_1 k_2 A_1^2 B_2}{\delta_1}\right)\left[\frac{2\exp\left(-A_1\sqrt{B_0}(\xi + c)\right)}{-\left(\frac{B_2}{B_0}\right) + \exp\left(-2A_1\sqrt{B_0}(\xi + c)\right)}\right]^2 \tag{3.52}$$

Where $\xi = k_1 x + k_2 t + \xi_0,$
In particular setting
$$k_1 = 1, k_2 = 1, \alpha_0 = 1, \xi_0 = 0, \delta_0 = 1, \delta_1 = 2, \delta_2 = \frac{1}{4}, c = 0, A_1 = 1, B_2 = 4$$
We find

$$u_{3,2}^0(\xi) = 1 + 24\frac{\exp(-4\xi)}{(\exp(-4\xi) - 1)^2}; \qquad \xi = x + t \qquad (See\ Figure\ 15) \tag{3.53}$$

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Substituting (3.45) and (3.49) into (3.2) we find

$$u_{3,3}(\xi)=\alpha_0+\left(\frac{12\delta_2 k_1 k_2 A_1^2 B_2}{\delta_1}\right)\left[\frac{-1}{\tan\left(A_1\sqrt{-B_0}(\xi+c)\right)\sqrt{\frac{-B_2}{B_0\left(\tan^2\left(A_1\sqrt{-B_0}(\xi+c)\right)+1\right)}}}\right]^2 \quad (3.54)$$

where $\xi=k_1x+k_2t+\xi_0$,

In particular setting

$$k_1=1,k_2=1,\alpha_0=1,\xi_0=\frac{\pi}{4},\delta_0=1,\delta_1=2,\delta_2=-\frac{1}{4},c=0,A_1=1,B_2=4$$

We find

$$u_{3,3}^0(\xi)=1-6\frac{\left(\cot^2\left(2\xi-\frac{\pi}{2}\right)+1\right)}{\cot^2\left(2\xi-\frac{\pi}{2}\right)};\qquad \xi=x+t+\frac{\pi}{4}\qquad (See\ Figure\ 16) \quad (3.55)$$

Case4:

$A_0=A_0,A_1=A_1,B_2=B_2,B_1=\frac{2A_0B_2}{A_1},B_0=B_0,k_1=k_1,k_2=k_2,\xi_0=\xi_0$

$\alpha_0=\left(\frac{4\delta_2k_2A_1^2k_1^2B_0+8\delta_2k_2k_1^2B_2A_0^2-\delta_0k_1-k_2}{k_1\delta_1}\right),\alpha_1=\left(\frac{24\delta_2B_2k_2k_1A_0A_1}{\delta_1}\right),$

$\alpha_2=\left(\frac{12\delta_2B_2k_2k_1A_1^2}{\delta_1}\right)$

(3.56)

Substituting (3.56) in to (2 .5), we have

$$\frac{d}{d\xi}F(\xi)=\left(A_0+A_1F(\xi)\right)\sqrt{B_0+\left(\frac{2A_0B_2}{A_1}\right)F(\xi)+B_2F^2(\xi)} \quad (3.57)$$

Using Maple, some solutions of Eq (3.57) are:

1) if $A_0=0,B_0=0,A_1\neq 0,B_2\neq 0$

$$F_{4,1}(\xi)=\frac{1}{\left(A_1\sqrt{B_2}(\xi+c)\right)} \quad (3.58)$$

2) if $A_0=0,B_2\succ 0,B_0\succ 0,$

$$F_{4,2}(\xi)=\frac{-2B_0\exp\left(-A_1\sqrt{B_0}(\xi+c)\right)}{B_2-B_0\exp\left(-2A_1\sqrt{B_0}(\xi+c)\right)} \quad (3.59)$$

3) if $A_0=0,B_2\succ 0,B_0\prec 0,$

$$F_{4,3}(\xi)=\frac{-B_0\left[1+\tan^2\left(A_1\sqrt{-B_0}(\xi+c)\right)\right]}{\tan\left(A_1\sqrt{-B_0}(\xi+c)\right)\sqrt{-B_0B_2\left[1+\tan^2\left(A_1\sqrt{-B_0}(\xi+c)\right)\right]}} \quad (3.60)$$

4) if $A_0\neq 0,B_2\neq 0,A_1\neq 0,(-A_0^2B_2+A_1^2B_0)=\Delta_2\succ 0,$

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$$F_{4,4}(\xi) = -\frac{2\sqrt{\Delta_2}\exp\left(-\sqrt{\Delta_2}(\xi+c)\right) - A_0\exp\left(-2\sqrt{\Delta_2}(\xi+c)\right) + A_0B_2}{A_1\left[B_2 - \exp\left(-2\sqrt{\Delta_2}(\xi+c)\right)\right]} \tag{3.61}$$

5) if $A_0 \neq 0, B_2 \neq 0, A_1 \neq 0, (-A_0^2B_2 + A_1^2B_0) = -\Delta_2 \prec 0$,

$$F_{4,5}(\xi) = -\frac{\Delta_2 + \Delta_2\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right) + A_0\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}}{A_1\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}} \tag{3.62}$$

6) if $A_0 \neq 0, B_2 \neq 0, A_1 \neq 0, (-A_0^2B_2 + A_1^2B_0) = 0$;

$$F_{4,6}(\xi) = \left[\frac{-1}{A_1\sqrt{B_2}(\xi+c)} - \frac{A_0}{A_1}\right] \tag{3.63}$$

Substituting (3.56) and (3.58) into (3.2) we find

$$u_{4,1}(\xi) = \left(\frac{4\delta_2k_2A_1^2k_1^2B_0 - \delta_0k_1 - k_2}{k_1\delta_1}\right) + \frac{\left(\frac{12\delta_2B_2k_2k_1A_1^2}{\delta_1}\right)}{\left(A_1\sqrt{B_2}(\xi+c)\right)^2} \tag{3.64}$$

Where $\xi = k_1x + k_2t + \xi_0$,

In particular setting

$k_1 = -1, k_2 = 1, \xi_0 = -1, \delta_0 = 2, \delta_1 = \delta_2 = 1, c = 1, A_1 = 1, A_0 = 0, B_0 = 0, B_2 = 4$

We find

$$u_{4,1}^0(\xi) = -1 - \frac{2}{(\xi+1)^2}; \qquad \xi = -x + t - 1 \qquad (\mathit{See\ Figure}\ 17) \tag{3.65}$$

Substituting (3.56) and (3.59) into (3.2) we find

$$u_{4,2}(\xi) = \left(\frac{4\delta_2k_2A_1^2k_1^2B_0 - \delta_0k_1 - k_2}{k_1\delta_1}\right) + \left(\frac{12\delta_2B_2k_2k_1A_1^2}{\delta_1}\right)\left[\frac{-2B_0\exp\left(-A_1\sqrt{B_0}(\xi+c)\right)}{B_2 - B_0\exp\left(-2A_1\sqrt{B_0}(\xi+c)\right)}\right]^2 \tag{3.66}$$

where $\xi = k_1x + k_2t + \xi_0$,

In particular setting

$k_1 = 1, k_2 = -1, \xi_0 = 0, \delta_0 = \delta_1 = \delta_2 = 1, c = 0, A_1 = 1, A_0 = 0, B_0 = 4$,
$B_2 = \exp(-25)$

We find

$$u_{4,2}^0(\xi) = -16 - \frac{768\exp(-25)\exp(-4\xi)}{\left(\exp(-25) - 4\exp(-4\xi)\right)^2}; \qquad \xi = x - t \qquad (\mathit{See\ Figure}\ 18) \tag{3.67}$$

Substituting (3.56) and (3.60) into (3.2) we find

$$u_{4,3}(\xi) = \left(\frac{4\delta_2k_2A_1^2k_1^2B_0 - \delta_0k_1 - k_2}{k_1\delta_1}\right) + \left(\frac{12\delta_2B_2k_2k_1A_1^2}{\delta_1}\right) \times \left(\frac{-B_0\left[1+\tan^2\left(A_1\sqrt{-B_0}(\xi+c)\right)\right]}{\tan\left(A_1\sqrt{-B_0}(\xi+c)\right)\sqrt{-B_0B_2\left[1+\tan^2\left(A_1\sqrt{-B_0}(\xi+c)\right)\right]}}\right)^2 \tag{3.68}$$

where $\xi = k_1x + k_2t + \xi_0$,

In particular setting

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$k_1 = 1, k_2 = -1, \xi_0 = 0, \delta_0 = \delta_1 = 1, \delta_2 = -2, c = 0, A_1 = 1, A_0 = 0, B_0 = -9, B_2 = 4$

We find

$$u_{4,3}^0(\xi) = -72 + 216\left(\frac{1+\tan^2(-3\xi)}{\tan^2(-3\xi)}\right); \qquad \xi = x - t \qquad (See\ Figure\ 19) \quad (3.69)$$

Substituting (3.56) and (3.61) into (3.2) we find

$$u_{4,4}(\xi) = \left(\frac{4\delta_2 k_2 A_1^2 k_1^2 B_0 + 8\delta_2 k_2 k_1^2 B_2 A_0^2 - \delta_0 k_1 - k_2}{k_1 \delta_1}\right) + \quad (3.70)$$
$$+\left(\frac{24\delta_2 B_2 k_2 k_1 A_0 A_1}{\delta_1}\right)\left[-\frac{2\sqrt{\Delta_2}\exp\left(-\sqrt{\Delta_2}(\xi+c)\right) - A_0\exp\left(-2\sqrt{\Delta_2}(\xi+c)\right) + A_0 B_2}{A_1\left[B_2 - \exp\left(-2\sqrt{\Delta_2}(\xi+c)\right)\right]}\right]$$
$$+\left(\frac{12\delta_2 B_2 k_2 k_1 A_1^2}{\delta_1}\right)\left[-\frac{2\sqrt{\Delta_2}\exp\left(-\sqrt{\Delta_2}(\xi+c)\right) - A_0\exp\left(-2\sqrt{\Delta_2}(\xi+c)\right) + A_0 B_2}{A_1\left[B_2 - \exp\left(-2\sqrt{\Delta_2}(\xi+c)\right)\right]}\right]^2$$

where $\xi = k_1 x + k_2 t + \xi_0$,

In particular setting

$k_1 = 1, k_2 = 1, \xi_0 = 0, \delta_0 = 1, \delta_1 = -1, \delta_2 = 2, c = 0, A_1 = 3, A_0 = 1, B_0 = 1, B_2 = 5$

We find

$$u_{4,4}^0(\xi) = -30\frac{(25+54\exp(-4\xi)+\exp(-8\xi))}{(\exp(-4\xi)-5)^2}; \qquad \xi = x + t \qquad (See\ Figure\ 20) \quad (3.71)$$

Substituting (3.56) and (3.62) into (3.2) we find

$$u_{4,5}(\xi) = \left(\frac{4\delta_2 k_2 A_1^2 k_1^2 B_0 + 8\delta_2 k_2 k_1^2 B_2 A_0^2 - \delta_0 k_1 - k_2}{k_1 \delta_1}\right) + \left(\frac{24\delta_2 B_2 k_2 k_1 A_0 A_1}{\delta_1}\right) \times \quad (3.72)$$
$$\left[-\frac{\Delta_2 + \Delta_2\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right) + A_0\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2 B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}}{A_1\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2 B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}}\right]$$
$$+\left(\frac{12\delta_2 B_2 k_2 k_1 A_1^2}{\delta_1}\right) \times$$
$$\left[-\frac{\Delta_2 + \Delta_2\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right) + A_0\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2 B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}}{A_1\tan\left(\sqrt{\Delta_2}(\xi+c)\right)\sqrt{\Delta_2 B_2\left(1+\tan^2\left(\sqrt{\Delta_2}(\xi+c)\right)\right)}}\right]^2$$

where $\xi = k_1 x + k_2 t + \xi_0$,

In particular setting

$k_1 = 1, k_2 = -1, \xi_0 = 0, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 0, A_1 = 2, A_0 = 2, B_0 = \frac{1}{4}, B_2 = 1$

We find

$$u_{4,5}^0(\xi) = -12\left(\frac{2\tan^2(\sqrt{3}\xi)+3}{\tan^2(\sqrt{3}\xi)}\right); \qquad \xi = x - t \qquad (See\ Figure\ 21) \quad (3.73)$$

Substituting (3.56) and (3.63) into (3.2) we find

$$u_{4,6}(\xi) = \left(\frac{4\delta_2 k_2 A_1^2 k_1^2 B_0 + 8\delta_2 k_2 k_1^2 B_2 A_0^2 - \delta_0 k_1 - k_2}{k_1 \delta_1}\right) + \quad (3.74)$$
$$+\left(\frac{24\delta_2 B_2 k_2 k_1 A_0 A_1}{\delta_1}\right)\left[\frac{-1}{A_1\sqrt{B_2}(\xi+c)} - \frac{A_0}{A_1}\right]$$

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$$+\left(\frac{12\delta_2 B_2 k_2 k_1 A_1^2}{\delta_1}\right)\left[\frac{-1}{A_1\sqrt{B_2}(\xi+c)}-\frac{A_0}{A_1}\right]^2$$

where $\xi = k_1 x + k_2 t + \xi_0$,

In particular setting
$k_1 = 1, k_2 = 1, \xi_0 = 2, \delta_0 = 1, \delta_1 = 1, \delta_2 = 1, c = 3, A_1 = 1, A_0 = 1, B_0 = 1, B_2 = 1$

We find

$$u_{4,6}^0(\xi) = -2\frac{(\xi^2+6\xi+3)}{(\xi+3)^2}; \qquad \xi = x + t + 2 \qquad (See\ Figure\ 22) \qquad (3.75)$$

4 Table Graphics

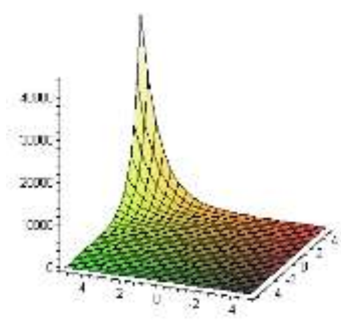	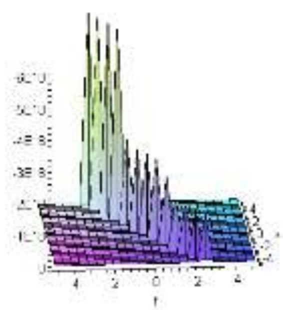	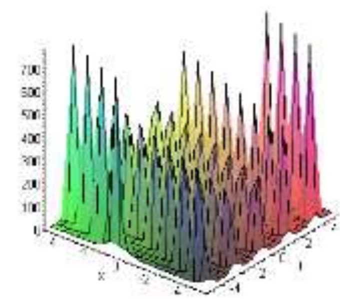
Figure 1 $u_{1,1}^0(\xi) = 1 + 2\exp(\xi)$ $\xi = x + t$	Figure 2 $u_{1,2}^0(\xi) = 1 - \frac{24}{\xi^2}$ $\xi = x - 2t$	Figure 3 $u_{1,3}^0(\xi) = 4 + 3\tan^2\left(-\frac{\xi}{2}\right)$ $\xi = x - 3t$

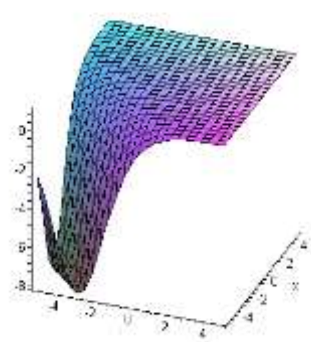	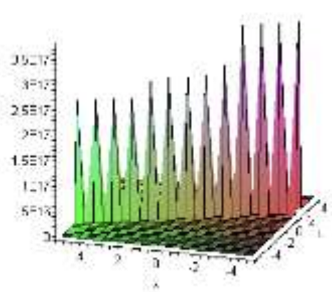	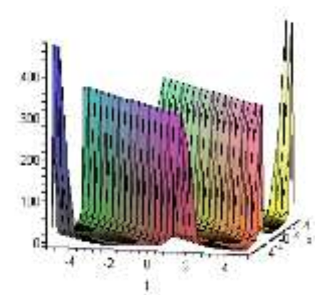
Figure 4 $u_{1,4}^0(\xi) = 1 - \frac{36\exp(\xi-1)}{(\exp(\xi-1)+1)^2}$ $\xi = x + t + 2^\pi$	Figure 5 $u_{2,1}^0(\xi) = -1 + \frac{24}{\xi^2}$ $\xi = x + 2t$	Figure 6 $u_{2,2}^0(\xi) = 3\tan^2\left(\frac{1}{2}\xi\right)$ $\xi = x + t$

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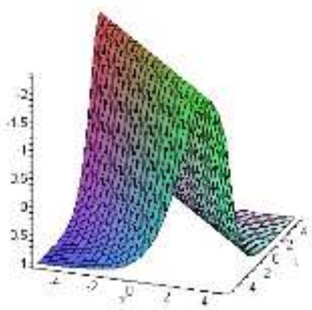

Figure 7
$u_{2,3}^{0}(\xi) = \left[1 - 36\frac{\exp(\xi-1)}{(\exp(\xi-1)+1)^2}\right]$
$\xi = x + t + 1$

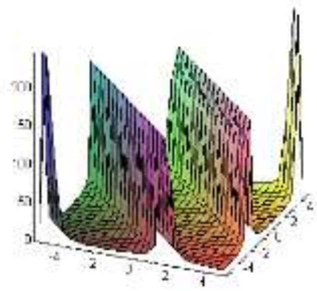

Figure 8
$u_{2,4}^{0}(\xi) = -1 + \frac{3}{2}\cot^2\left(\frac{1}{2}\xi\right)$
$\xi = x + t + \pi$

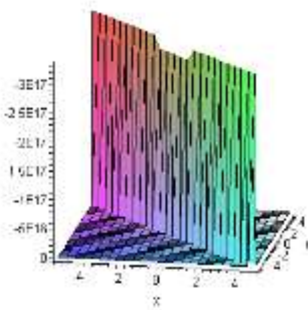

Figure 9
$u_{2,5}^{0}(\xi) = \left[2 - 3\left(\frac{\exp(-\xi)+1}{\exp(-\xi)-1}\right)^2\right]$
$\xi = x + t$

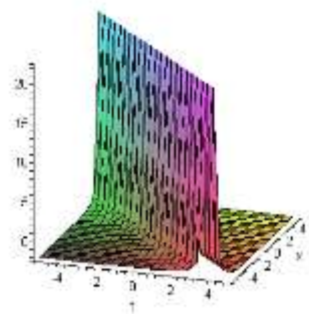

Figure 10
$u_{2,6}^{0}(\xi) = 1 - \frac{18(\xi+1)^2-4}{6(\xi+1)^2}$
$\xi = x + t$

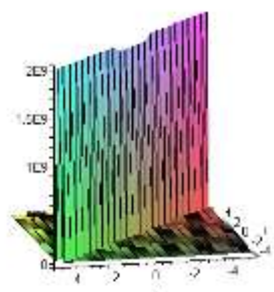

Figure 11
$u_{2,7}^{0}(\xi) = 4 - 6\frac{(\xi^2-2)}{\xi}$
$\xi = x + t$

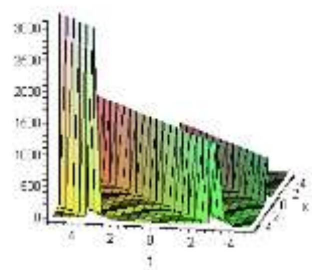

Figure 12
$u_{2,8}^{0}(\xi) = 3\tan^2\left(\frac{\xi}{2}\right)$
$\xi = x + t + \frac{\pi}{2}$

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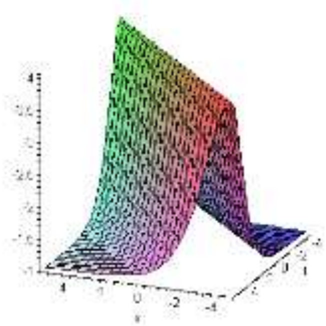

Figure 13

$$u_{2,9}^{0}(\xi) = 2 - 3\frac{[(\exp(\xi)-1)^2 + 8\exp(\xi)]}{[\exp(\xi)+1]^2}$$
$$\xi = x + t$$

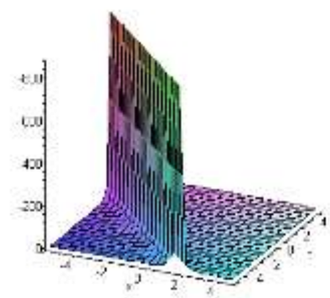

Figure 14

$$u_{1,3}^{0}(\xi) = 1 - \frac{6}{\xi^2}$$
$$\xi = x + t + 3$$

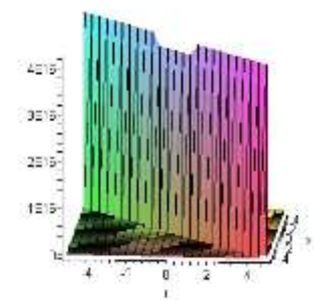

Figure 15

$$u_{2,3}^{0}(\xi) = 1 + 24\frac{\exp(-4\xi)}{(\exp(-4\xi)-1)^2}$$
$$\xi = x + t$$

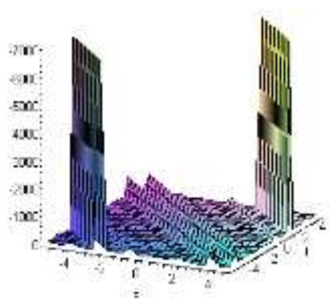

Figure 16

$$u_{3,3}^{0}(\xi) = 1 - 6\frac{\left(\cot^2\left(2\xi - \frac{\pi}{2}\right) + 1\right)}{\cot^2\left(2\xi - \frac{\pi}{2}\right)}$$
$$\xi = x + t + \frac{\pi}{4}$$

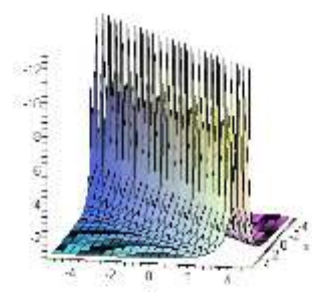

Figure 17

$$u_{1,4}^{0}(\xi) = -1 - \frac{2}{(\xi+1)^2}$$
$$\xi = -x + t - 1$$

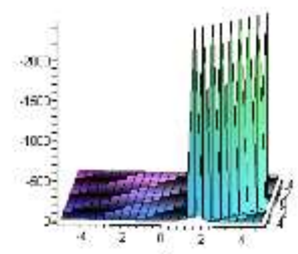

Figure 18

$$u_{2,4}^{0}(\xi) = -16 - \frac{768\exp(-25)\exp(-4\xi)}{\left(\exp(-25) - 4\exp(-4\xi)\right)^2}$$
$$\xi = x - t$$

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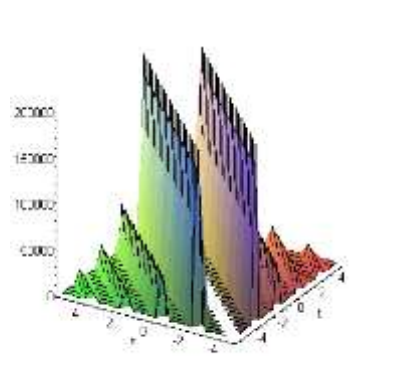	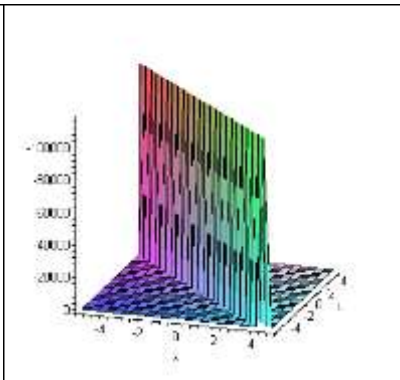	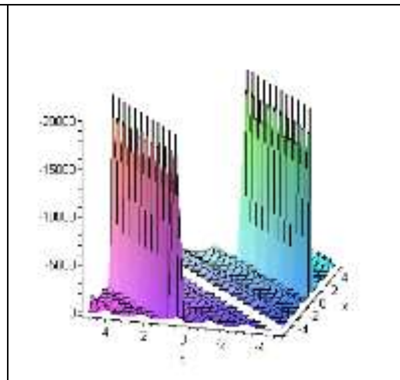
Figure 19 $u_{3,4}^{0}(\xi) = -72 + 216\left(\frac{1+\tan^2(-3\xi)}{\tan^2(-3\xi)}\right)$ $\xi = x - t$	Figure 20 $u_{4,4}^{0}(\xi) = -30\frac{(25+54\exp(-4\xi)+\exp(-8\xi))}{(\exp(-4\xi)-5)^2}$ $\xi = x + t$	Figure 21 $u_{5,4}^{0}(\xi) = -12\left(\frac{2\tan^2(\sqrt{3}\xi)+3}{\tan^2(\sqrt{3}\xi)}\right)$ $\xi = x - t$

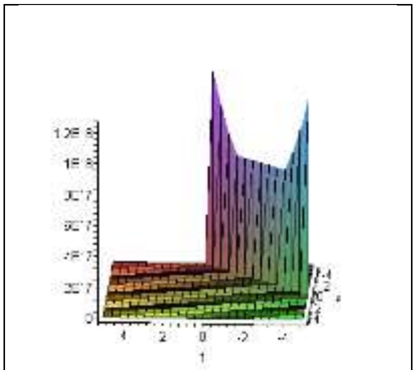

Figure 22
$u_{6,4}^{0}(\xi) = -2\frac{(\xi^2+6\xi+3)}{(\xi+3)^2}$
$\xi = x + t + 2$

CONCLUSION

In this article, we have proposed a new method called calibrating auxiliary differential equation method using the generalized wave transformation (2.2), to obtain the exact solutions for BBM equation. The main advantage of this method is its capability of greatly reducing the size of

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computational work compared to existing techniques.
The method could be used for a large class of very interesting nonlinear equations.

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