BLOCK IMPLICIT ONE-STEP METHOD FOR THE NUMERICAL INTEGRATION OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: *In this paper, block implicit one-step method of order seven is proposed for the numerical integration of first order initial value problems. The method is based on collocation of the differential system and interpolation of the approximate at the grid and off-grid points. The procedure yields six consistent finite difference schemes which are combined as simultaneous numerical integrators to form block method. The method is found to be zero-stable hence convergent. The accuracy of the method is tested with some standard first order initial value problems. The results show a better performance over the existing methods*.

KEYWORDS: Block, Implicit, Finite Differential Schemes, Simultaneous Numerical Integrators, Zero-Stable And Convergent.

2010 Mathematics Subject Classifications:65L05,65L06,65L07 and 65L20

INTRODUCTION

Many problems encountered in the various branches of science, engineering and management give rise to differential equations of the form:

$$y' = f(x, y), y(x_0) = y_0, a \le x \le b \qquad (1.1)$$

where f is assumed to be Lipschiz constants.

The solution of (1.1) has been discussed by various researchers among them are [see Lie and Norsett (1989), Onumanyi et al. (1994, 1999, 2002), Sirisena [(1999, 2004), Lambert (1973) and Gear (1971)]. However, experience has shown in [Lie and Norsett (1989), and Onumanyi et al. (1994)] that the traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation. These earlier works have focused on the construction of continuous multistep methods by employing the multistep collocation. The continuous multistep methods produce piecewise polynomial solutions over k-steps $[x_n, x_{n+k}]$ for the first order systems of ordinary differential equation (ODEs). Sirisena et al. (2004) developed a continuous new Butcher type two-step block hybrid multistep method for problem (1.1). The results obtained showed a class of discrete schemes of order 5 and error constants ranging from $C_6 = 1.45\times10^{-5}$ to $C_6 = 1.790\times10^{-4}$. In Areo et al. (2009), we reported one-step embedded Butcher type two-step block hybrid schemes employing basis functions as approximate and more recently we proposed sixth-order hybrid block method for the numerical solution of first order initial value problems, see Areo et al. (2013) , but in this paper effort is being made to extend the scope. In this paper, we propose block implicit one-step method for the numerical integration of initial value problems in ordinary differential equations.

The Derivation of the Method

In this section, the derivation of the continuous formulation of the proposed block implicit one-step method for the numerical integration of initial value problems in ordinary differential equations is presented and employs it to deduce the discrete ones. The continuous scheme is used to obtain finite difference methods which are combined as simultaneous numerical integrators to constitute conveniently the block method.

In order to derive the continuous scheme, the method of Sirisena et al. (2004) is applied where a k-step multistep collocation method with m collocation points was obtained as follows:

$$\bar{y}(x) = \sum_{j=0}^{t-1}\alpha_j(x)y(x_{n+j}) + h\sum_{j=0}^{t-1}\beta_j(x)f(\bar{x}_j, \bar{y}(\bar{x}_j)) \tag{2.1}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are the continuous coefficients of the method. Where $\alpha_j(x)$ and $\beta_j(x)$ are defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1}\alpha_{j,i+1}x^j; \quad j \in \{0,1,...,t-1\} \tag{2.2}$$

and

$$\beta_j(x) = \sum_{i-1}^{t+m-1}\beta_{j,i+1}x^j; \quad j \in \{0,1,2,...,t-1\} \tag{2.3}$$

x_{n+j} : $j = 0,1,2,...,t-1$ in (2.1) are $(0 \le t \le k)$ arbitrary chosen interpolation points taken from $\{x_n,...,x_{n+k}\}$ and $\bar{x}_j$: $j = 0,1,...,m-1$ are the m collocation points belonging to $\{x_n,...,x_{n+k}\}$. To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena et al. (2004) arrived at a matrix equation of the form

$$DC = I \tag{2.4}$$

Where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{t+m-2} \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{t+m-2} \end{pmatrix} \tag{2.5}$$

The above matrix (2.5) is the multistep collocation matrix of dimension $(t+m)\times(t+m)$ and

$$C = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{pmatrix} \tag{2.6}$$

Where t and m are defined as the number of interpolation points and the number of collocation points used respectively. The columns of the matrix $C = D^{-1}$ give the continuous coefficients

$$\alpha_j(x); \quad j = 0,1,...,k-1 \quad and \quad \beta_j(x); \quad j = 0,1,...,k-1$$

The proposed sixth-order hybrid block method was developed subjected to the following conditions for matrix D:

$$\{t=1, x_0 = x_n, x_1 = x_{n+1/4}\}\ \{m=7, \bar{x}_n = x_n, \bar{x}_1 = x_{n+1/4}, \bar{x}_2 = x_{n+1/2}, \bar{x}_3 = x_{n+3/4}, \bar{x}_{7/2} = x_{n+7/8}, \bar{x}_{15/8} = x_{n+15/16}, \bar{x}_4 = x_{n+1}\} \qquad (2.7)$$

and (2.1) becomes

$$\bar{y}(x) = \alpha_0(x) y_n + \alpha_{1/4}(x) y_{n+1/4} + h\left[\beta_0(x) f_n + \beta_{1/4}(x) f_{n+1/4} + \beta_{1/1}(x) f_{n+1/2} + \beta_{3/4}(x) f_{n+3/4} + \beta_{7/8}(x) f_{n+7/8} + \beta_1(x) f_{n+1}\right]$$

Thus the matrix D in (2.5) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1/4} & x_{n+1/4}^2 & x_{n+1/4}^3 & x_{n+1/4}^4 & x_{n+1/4}^5 & x_{n+1/4}^6 & x_{n+1/4}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1/4} & 3x_{n+1/4}^2 & 4x_{n+1/4}^3 & 5x_{n+1/4}^4 & 6x_{n+1/4}^5 & 7x_{n+1/4}^6 \\ 0 & 1 & 2x_{n+1/2} & 3x_{n+1/2}^2 & 4x_{n+1/2}^3 & 5x_{n+1/2}^4 & 6x_{n=1/2}^5 & 7x_{n+1/2}^6 \\ 0 & 1 & 2x_{n=3/4} & 3x_{n+3/4}^2 & 4x_{n=3/4}^3 & 5x_{n+3/4}^4 & 6x_{n+3/4}^5 & 7x_{n+3/4}^6 \\ 0 & 1 & 2x_{n+7/8} & 3x_{n+7/8}^2 & 4x_{n+7/8}^3 & 5x_{n=7/8}^4 & 6x_{n+7/8}^5 & 7x_{n+7/8}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n=1}^2 & 4x_{n+1}^3 & 5x_{n=1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \end{bmatrix} \qquad (2.8)$$

Thus, the elements of $C = D^{-1}$ were obtained such that $C = (c_{i,j})$, $1 \le i, j \le 8$

From (2.2) and (2.3) using the elements of $C = D^{-1}$ we have,

$$\alpha_0(x) = \frac{1}{1159h^7}[393216(x-x_n)^7 - 1548288(x-x_n)^6 h + 2408448(x-x_n)^5 h^2 - 1854720(x-x_n)^4 h^3 + 713216(x-x_n)^3 h^4 - 112896(x-x_n)^2 h^5 + 1159h^7] \qquad (2.9)$$

$$\alpha_{1/4}(x) = \frac{1}{1159h^7}[-393216(x-x_n)^7 + 1548288(x-x_n)^6 h - 2408448(x-x_n)^5 h^2 + 1854720(x-x_n)^4 h^3 - 713216(x-x_n)^3 h^4 + 112896(x-x_n)^2 h^5] \qquad (2.10)$$

$$\beta_0(x) = \frac{1}{730170h^6}[19931136(x-x_n)^7 - 79962368(x-x_n)^6 h + 128086464(x-x_n)^5 h^2 - 103746720(x-x_n)^4 h^3 + 44148236(x-x_n)^3 h^4 - 9182031(x-x_n)^2 h^5 + 730170(x-x_n)h^6] \qquad (2.11)$$

$$\beta_{1/4}(x) = \frac{1}{52155h^6}[4632576(x-x_n)^7 - 17647360(x-x_n)^6 h + 26149248(x-x_n)^5 h^2 - 18652080(x-x_n)^4 h^3 + 6270016(x-x_n)^3 h^4 - 745920(x-x_n)^2 h^5] \qquad (2.12)$$

$$\beta_{1/2}(x) = \frac{1}{52155h^6}[-3280896(x-x_n)^7 + 11435008(x-x_n)^6 h - 14977344(x-x_n)^5 h^2 + 8938560(x-x_n)^4 h^3 - 2311636(x-x_n)^3 h^4 + 211806(x-x_n)^2 h^5] \qquad (2.13)$$

$$\beta_{3/4}(x) = \frac{1}{2745h^6}[208896(x-x_n)^7 - 666368(x-x_n)^6 h + 787584(x-x_n)^5 h^2 - 421680(x-x_n)^4 h^3 - 8843264(x-x_n)^3 h^4 + 755712(x-x_n)^2 h^5] \qquad (2.14)$$

$$\beta_{7/8}(x) = \frac{1}{365085h^6}[-21233664(x-x_n)^7 + 64618496(x-x_n)^6 h - 73089024(x-x_n)^5 h^2 + 37847040(x-x_n)^4 h^3$$

$$-8843264(x-x_n)^3 h^4 + 755712(x-x_n)^2 h^5] \qquad (2.15)$$

$$\beta_1(x) = \frac{1}{104310h^6}[1425408(x-x_n)^7 - 4129024(x-x_n)^6 h + 4502592(x-x_n)^5 h^2 - 2272800(x-x_n)^4 h^3$$
$$+522388(x-x_n)^3 h^4 - 44163(x-x_n)^2 h^5] \qquad (2.16)$$

On substituting equations (2.9)-(2.16) the above into (2.7), we obtained the continuous scheme as follows:

$$\bar{y}(x) = \frac{1}{1159h^7}[393216(x-x_n)^7 - 1548288(x-x_n)^6 h + 2408448(x-x_n)^5 h^2 - 1854720(x-x_n)^4 h^3$$
$$+713216(x-x_n)^3 h^4 - 112896(x-x_n)^2 h^5 + 1159h^7]\, y_n$$
$$+\frac{1}{1159h^7}[-393216(x-x_n)^7 + 1548288(x-x_n)^6 h - 2408448(x-x_n)^5 h^2 + 1854720(x-x_n)^4 h^3$$
$$-713216(x-x_n)^3 h^4 + 112896(x-x_n)^2 h^5]\, y_{n+1/4}$$
$$+\frac{1}{730170h^6}[19931136(x-x_n)^7 - 79962368(x-x_n)^6 h + 128086464(x-x_n)^5 h^2 - 103746720(x-x_n)^4 h^3$$
$$+44148236(x-x_n)^3 h^4 - 9182031(x-x_n)^2 h^5 + 730170(x-x_n)h^6]\, f_n$$
$$+\frac{1}{52155h^6}[4632576(x-x_n)^7 - 17647360(x-x_n)^6 h + 26149248(x-x_n)^5 h^2 - 18652080(x-x_n)^4 h^3$$
$$+6270016(x-x_n)^3 h^4 - 745920(x-x_n)^2 h^5]\, f_{n+1/4}$$
$$+\frac{1}{52155h^6}[-3280896(x-x_n)^7 + 11435008(x-x_n)^6 h - 14977344(x-x_n)^5 h^2 + 8938560(x-x_n)^4 h^3$$
$$-2311636(x-x_n)^3 h^4 + 211806(x-x_n)^2 h^5]\, f_{n+1/2}$$
$$+\frac{1}{2745h^6}[208896(x-x_n)^7 - 666368(x-x_n)^6 h + 787584(x-x_n)^5 h^2 - 421680(x-x_n)^4 h^3$$
$$-2311636(x-x_n)^3 h^4 + 211806(x-x_n)^2 h^5]\, f_{n+3/4}$$
$$+\frac{1}{365085h^6}[-21233664(x-x_n)^7 + 64618496(x-x_n)^6 h - 73089024(x-x_n)^5 h^2 + 37847040(x-x_n)^4 h^3$$
$$-8843264(x-x_n)^3 h^4 + 755712(x-x_n)^2 h^5]\, f_{n+7/8}$$
$$+\frac{1}{104310h^6}[1425408(x-x_n)^7 - 4129024(x-x_n)^6 h + 4502592(x-x_n)^5 h^2 - 2272800(x-x_n)^4 h^3$$
$$+522388(x-x_n)^3 h^4 - 44163(x-x_n)^2 h^5]\, f_{n+1} \qquad (2.17)$$

Now, evaluating (2.17) at $x = x_{n+1}, x = x_{n+15/16}, x = x_{n+7/8}, x = x_{n+3/4}, x = x_{n+1/2}$ and its 1st derivative at $x = x_{n+15/16}$, we obtain the following six discrete schemes which constitute the block method:

$$y_{n+1}-\frac{135}{1159}y_n-\frac{1024}{1159}y_{n+1/4}=\frac{h}{81130}[543f_n+10080f_{n+1/4}+24108f_{n+1/2}+12768f_{n+3/4}+12288f_{n+7/8}+3423f_{n+1}] \tag{2.18}$$

$$y_{n+15/16}-\frac{42229}{296704}y_n-\frac{254475}{296704}y_{n+1/4}=\frac{h}{1063387136}[9182085f_n+140424900f_{n+1/4}+306622470f_{n+1/2}$$
$$+192150420f_{n+3/4}+114069120f_{n+7/8}+6466845f_{n+1}] \tag{2.19}$$

$$y_{n+7/8}-\frac{5875}{37088}y_n-\frac{31213}{37088}y_{n+1/4}=\frac{h}{21362688}[210175f_n+2925790f_{n+1/4}+6036800f_{n+1/2}+4236050f_{n+3/4}$$
$$+797440f_{n+7/8}-8575f_{n+1}] \tag{2.20}$$

$$y_{n+3/4}-\frac{160}{1159}y_n-\frac{999}{1159}y_{n+1/4}=\frac{h}{324520}[2703f_n+42378f_{n+1/4}+94458f_{n+1/2}+40698f_{n+3/4}-7680f_{n+7/8}+903f_{n+1}] \tag{2.21}$$

$$y_{n+1/2}-\frac{311}{1159}y_n-\frac{848}{1159}y_{n+1/4}=\frac{h}{2920680}[51755f_n+511448f_{n+1/4}+428680f_{n+1/2}-122360f_{n+3/4}$$
$$+69632f_{n+7/8}-13055f_{n+1}] \tag{2.22}$$

$$\frac{2816568315}{593408}y_{n+1/4}-\frac{2816568315}{593408}y_n=\frac{h}{299077632}[-10999296f_n-44538795f_{n+1/4}+51295167f_{n+1/2}$$
$$-145350513f_{n+3/4}+333571392f_{n+7/8}-299077632f_{n+15/16}+78426117f_{n+1}] \tag{2.23}$$

THE BASIC PROPERTIES OF THE METHOD

Order, Error Constant and Consistency of the Method

The six finite difference schemes (2.18)-(2.23) derived are discrete schemes belonging to the class of Linear Multistep Method (LMM) of the form

$$\sum_{j=0}^{k}\alpha_j(x)y(x_{n+j})=h\sum_{j=0}^{k}\beta_j(x)f(x_{n+j}). \tag{3.1}$$

This is a method associated with a linear difference operator,

$$L[y(x);h]=\sum_{j=0}^{k}(\alpha_j y(x+jh)=h\beta_j y''(x+jh)) \tag{3.2}$$

where $y(x)$ is an arbitrary function continuously differentiable on the interval $[a,b]$. The Taylor series expansion about the point x,

$$L[y(x);h]=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+\cdots+c_qh^qy^{(q)}(x), \tag{3.3}$$

where

$$C_0=\alpha_0+\alpha_1+\ldots+\alpha_k$$
$$C_1=(\alpha_0+\alpha_1+\ldots+\alpha_k)-(\beta_0+\beta_1+\ldots+\beta_k)$$
$$\vdots$$
$$C_q=\frac{1}{q!}(\alpha_1+2^q\alpha_2+\ldots+k^q\alpha_k)-\frac{1}{(q-1)!}(\beta_1+2^{q-1}\beta_2+\ldots+k^{q-1}\beta_k),q=2,3\cdots \tag{3.4}$$

Definition 3.1: The method (3.1) is said to be of order P if $C_0 = C_1 = C_2 = \ldots = C_p = 0$ and $C_{p+1} \neq 0$ is the error constant, see Lambert (1973). Applying this definition to equations (2.18)-(2.23) which make up the block method, it is verified that each of the six difference schemes is of order $p = (7,7,7,7,7,7)^T$ with error constants

$$\left(\frac{-1}{664616960}, \frac{1557875}{4460168918073344}, \frac{96775}{67201205796864}, \frac{1}{4253548544}, \frac{33411}{574229053440}, \frac{-169895}{4977867096064}\right)^T.$$

Definition 3.2: A LMM of the form (3.1) is said to be consistent if the LMM is of order $p \geq 1$. Since the discrete schemes derived in (2.18)-(2.23) are of order ≥ 1 according to Definition 3.2, therefore, the schemes are consistent.

Zero-Stability and Convergence of the Method

It is known from the literature that the stability of a LMM determines the manner in which the error is propagated as the the numerical computation proceeds. Hence, the investigation of the zero-stability property is necessary.

Definition 3.3: According to Lambert (1973), The LMM is said to be zero - stable if no root of the first characteristic polynomial $\rho(\xi)$ has modulus greater than one, and if every root with modulus one is

simple, where $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$. The investigation carried out on the six difference schemes in

(2.18)-(2.23) revealed that all the roots of the derived schemes are less than or equal to 1; hence the schemes are zero-stable. Since the consistency and zero-stable of the schemes (2.18)-(2.23) have been established, then the proposed hybrid block method is convergent, see Lambert (1973) and Fatunla (1988).

Numerical Experiment

In this section, the concern is the application of the schemes derived in section two in block form on some initial value problems with test problems 4.1.1-4.1.3 and an application problem 4.1.4:

Problems

Problem 4.1.1:

$$y' = -y; \quad y(0) = 1, \quad h = 0.1, 0 \leq x \leq 1 \ \ and \ \ y(x) = e^{-x}$$

[see Sirisena et al. (1999 and 2004) and Areo et al. (2009 and 2013)]

Problem 4.1.2:

$$y' = -8(y - x) + 1; y(0) = 2, h = 0.1, 0 \leq x \leq 1$$
$$y(x) = x + 2\ell^{-8x}$$

[see Sirisena et al. (1999 and 2004) and Areo et al. (2009 and 2013)]

Problem 4.1.3:

$$y' = x - y, y(0) = 0, h = 0.1, 0 \leq x \leq 1$$
$$y(x) = x + \ell^{-x} - 1$$

[see Sirisena et al. (1999 and 2004) and Areo et al. (2009 and 2013)]

Problem 4.1.4: Considering the discharge valve on a 200-gallon tank that is full of water opened at time $t = 0$ and 3 gallons per second flow out. At the same time 2 gallons per second of 1 percent chlorine mixture begin to enter the tank. Assume that the liquid is being stirred so that the concentration of chlorine is consistent throughout the tank. The task is to determine the concentration of chlorine when the tank is half full. It takes 100 seconds for this moment to occur, since we lose a gallon per second. If $y(t)$ is the amount of chlorine in the tank at time t, then the rate chlorine is entering is $\frac{2}{100}$ gal/sec and it is leaving at the rate $3[\frac{y}{200-t}]$ gal/sec.

Thus, the resulting IVP is

$$\frac{dy}{dt}=\frac{2}{100}-3\frac{y}{200-t},\; 0\le t\le 1:\; y(0)=0,\; h=0.1$$

whose analytical solution is

$$y(t)=2-\frac{1}{100}t-2[1-\frac{5t}{1000}]^3.$$

[See John L. Van Iwaarden (1985) and Areo et al. (2013)]

RESULTS

The comparison of errors for problems 4.1.1-4.1.4 are shown in the tables below.

Table 1: Comparison of absolute errors for **Problem 4.1.1**

X	Sirisena et al. (2004)	Areo et al. (2009)	Areo et al. (2013)	Proposed Method
0.1	2.00×10^{-9}	3.60×10^{-10}	0.0	0.0
0.2	2.00×10^{-9}	1.80×10^{-10}	0.0	0.0
0.3	1.00×10^{-9}	5.80×10^{-10}	0.0	0.0
0.4	2.00×10^{-9}	7.40×10^{-10}	0.0	0.0
0.5	1.00×10^{-9}	8.10×10^{-10}	0.0	0.0
0.6	3.00×10^{-9}	9.90×10^{-10}	1.00×10^{-10}	1×10^{-10}
0.7	2.00×10^{-9}	9.90×10^{-10}	0.0	0.0
0.8	3.00×10^{-9}	1.00×10^{-9}	1.00×10^{-10}	1×10^{-10}
0.9	3.00×10^{-9}	1.10×10^{-9}	0.0	0.0
1.0	3.00×10^{-9}	1.20×10^{-9}	1.00×10^{-10}	1×10^{-10}

Table 2: Comparison of absolute errors for **Problem 4.1.2**

X	Sirisena et al. (2004)	Areo et al. (2009)	Areo et al. (2013)	Proposed Method
0.1	3.60×10^{-4}	7.20×10^{-6}	1.99×10^{-7}	5.70×10^{-9}
0.2	1.50×10^{-4}	6.50×10^{-6}	1.79×10^{-7}	5.30×10^{-9}
0.3	5.90×10^{-5}	4.40×10^{-6}	1.20×10^{-8}	3.00×10^{-9}
0.4	1.60×10^{-5}	2.60×10^{-6}	7.23×10^{-8}	2.20×10^{-9}
0.5	4.30×10^{-5}	1.50×10^{-6}	3.98×10^{-8}	4.00×10^{-10}
0.6	2.10×10^{-5}	8.00×10^{-7}	2.80×10^{-8}	5.00×10^{-10}
0.7	5.70×10^{-7}	4.20×10^{-7}	1.10×10^{-8}	1.00×10^{-10}
0.8	1.60×10^{-6}	2.10×10^{-7}	5.30×10^{-9}	4.00×10^{-10}
0.9	5.10×10^{-6}	1.10×10^{-7}	2.30×10^{-9}	5.00×10^{-10}
1.0	2.80×10^{-6}	5.30×10^{-8}	1.00×10^{-9}	0.0

Table 3: Comparison of absolute errors for **Problem 4.1.3**

X	Sirisena et al. (2004)	Areo et al. (2009)	Areo et al. (2013)	Proposed Method
0.1	2.00×10^{-9}	3.80×10^{-11}	0.0	0.0
0.2	2.10×10^{-9}	7.80×10^{-11}	0.0	0.0
0.3	1.70×10^{-9}	1.00×10^{-10}	6.00×10^{-10}	$6.0 \text{ x } 10^{-10}$
0.4	0.00	1.30×10^{-10}	3.00×10^{-11}	$3.0 \text{ x } 10^{-11}$
0.5	6.70×10^{-9}	2.10×10^{-10}	0.0	0.0
0.6	0.00	1.90×10^{-10}	1.00×10^{-10}	$1.0 \text{ x } 10^{-10}$
0.7	1.00×10^{-9}	1.90×10^{-10}	0.0	0.0
0.8	0.00	2.20×10^{-10}	0.0	0.0
0.9	0.00	2.40×10^{-10}	0.0	0.0
1.0	0.00	2.70×10^{-10}	1.00×10^{-10}	$1.0 \text{ x } 10^{-10}$

Table 4: Comparison of absolute errors for **Problem 4.1.4**

X	Areo et al. (2013)	Proposed Method
0.1	0.0	0.0
0.2	0.0	0.0
0.3	2.40×10^{-11}	0.0
0.4	2.40×10^{-11}	0.0
0.5	2.40×10^{-11}	0.0
0.6	3.00×10^{-11}	0.0
0.7	3.00×10^{-11}	0.0
0.8	3.00×10^{-11}	0.0
0.9	3.00×10^{-11}	0.0
1.0	3.00×10^{-11}	0.0

CONCLUSION

A collocation approach which produces a family of order seven multiderivative schemes has been proposed for the numerical integration of initial value problems in ordinary differential equations. The errors arising from Problems 4.1.1-4.1.3 using the proposed method were compared with those obtained by Sirisena et al. (2004), Areo et l. (2009) and Areo et al. (2013) respectively, who earlier solved the same problems while the errors arising from Problem 4.1.4 were compared with Areo (2013).

A close look at the tables presented above reveal that the newly proposed method perform better than those compared with. The method is also desirable by virtue of possessing high order of accuracy.

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