

BINOMIAL MIXTURE OF ERLANG DISTRIBUTION**Kareema Abed Al-Kadim and Rusul Nasir AL-Hussani**

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ABSTRACT: *In this paper we introduce new Binomial mixture distribution, Binomial-Erlang Distribution (B-Er) based on the transformation $p = e^{-\lambda}$ where p on the interval $[0, \infty)$ by using Moments methods and Laplace transform in mixing binomial distribution with Erlang distribution.*

KEYWORDS: Binomial Distribution, Erlang Distribution, Mixture Distribution, Laplace transform.

INTRODUCTION

Since 1894 the concept of mixture distribution was studied by a number of authors as Blischke [2] who defined mixture distribution as a weighted average of probability distribution with positive weights, which were probability distributions, called the mixing distributions that sum to one.

And the finite mixture distribution arises, at the end of the last century, in a variety of applications ranging from the length of fish to the content of DNA in the nuclei of liver cells when Karl Pearson published his well-known paper on estimation the five parameters in mixture of two normal distribution. There are many authors presented some of these distributions like Kent John [7] (1983), Nassar and Mahmoud [12] (1988), Gleser L., [8] (1989), Jiang S. and Kececioglu D. [15] (1992), Sum and Oommen [17] (1995), Sultan, Ismail, and Al-Moisheer [16] (2007), Shawky and Bakoban [14], Mahir and Ali [10] (2009), Hanaa, Abu-Zinadah [5] (2010) Eri.o.lu, Ulku, Eri.o.lu, M., and Erol [3] (2011), Mubarak [11] (2011), Gómez-Déniz, Péreionz-Sánchez, Vázquez-Polo and Hernández-Bastide [4] (finite mixtures of simple distributions as statistical models, mixtures of exponential, gamma distribution with arbitrary scale parameter and shape parameter as a scale mixture of exponential, Weibull, model of two inverse Weibull, exponentiated gamma, exponentiated Pareto and exponential, Exponential-Gamma, Exponential-Weibull and Gamma-Weibull, Frechet Negative Binomial - Confluent hypergeometric) distributions.

And there are some other authors who used a transformation of parameter of Negative Binomial $p = e^{-\lambda}$ or $p = 1 - e^{-\lambda}$, or they used together in construction the mixture distribution as follows:

Zamani and Ismail (2010) [18] introduced a mixed distribution, Negative Binomial - Lindley distribution based on the transformation of parameter of Negative Binomial $p = e^{-\lambda}$ and derivation of its factorial moment, maximum likelihood and moment estimation of its parameters with application.

Lord (2011) [9] introduced a mixed distribution, Negative Binomial - Lindley distribution where the transformation of $p = 1 - e^{-\lambda}$

Irungu(2011) [6] constructed some Negative Binomial mixtures generated by randomizing the success parameter p and fixing parameter r reparameterization of $p = e^{-\lambda}$ and $p = 1 - e^{-\lambda}$ of a Negative Binomial Distribution. The mixing distributions used are Exponential, Gamma, Exponentiated Exponential, Beta Exponential, Variate Gamma, Variate Exponential, Inverse Gaussian, and Lindely with some of their properties.

Pudprommarat, Bodhisuwan and Zeephongsekul[13] (2012) introduced a Negative Binomial -Beta Exponential distribution with the same one assumption $p = e^{-\lambda}$ with derivation its factorial moments, moments of order statistics and the maximum likelihood estimation of its parameters.

Aryuyuen,S. and Bodhisuwan, W. (2013), [1] (2013) introduced a new mixed distribution, the Negative Binomial-Generalized Exponential (NB-GE) distribution based on transformation $p = e^{-\lambda}$ with some of its properties and estimation of its parameter.

In this paper we introduce new Binomial mixture distribution, Binomial-Erlang Distribution(B-Er) based the transformation

$p = e^{-\lambda}, \lambda > 0$ on the interval $[0, \infty)$, with some of their properties using Moments methods and Laplace transform in mixing binomial distribution with Erlang distribution.

Method of mixing binomial distributions with other distributions

Method of moments:

According [6], we can define the p.m.f as

$$\begin{aligned} f(x) &= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} g(p) dp \\ &= \binom{n}{k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{j-k} \int_0^1 p^{k+r} g(p) dp \\ &= \binom{n}{k} \sum_{r=0}^{n-k} (-1)^{j-k} \binom{n-k}{j-k} \int_0^1 p^j g(p) dp \\ &= \binom{n}{k} \sum_{r=0}^{n-k} (-1)^{j-k} \binom{n-k}{j-k} E(p^j) \\ f(x = k) &= \begin{cases} \sum_{r=0}^{n-k} \frac{(-1)^{j-k} n!}{k!(j-k)!(n-j)!} E(p^j) I_{(0,\infty)}(x) & j \geq k \\ 0 & j < k \end{cases} \quad (1) \end{aligned}$$

Where $E(p^j)$ is the j th moment about the origin of the mixing distribution with other distribution .

Method of Laplace transform:

According [6], with assumption that $p = e^{-\lambda}$, we can define the p.m.f as

$$f(x = k) = \binom{n}{k} \int_0^{\infty} p^k (1-p)^{n-k} g(p) dp$$

$$\begin{aligned}
 &= \binom{n}{k} \int_0^\infty e^{-\lambda k} (1 - e^{-\lambda})^{n-k} g(\lambda) d\lambda \\
 &= \binom{n}{k} \int_0^\infty e^{-\lambda k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} e^{-\lambda r} g(\lambda) d\lambda \\
 &= \binom{n}{k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \int_0^\infty e^{-\lambda(k+r)} g(\lambda) d\lambda
 \end{aligned}$$

Where $L_\lambda(k+r) = E(e^{-\lambda(k+r)}) = \int_0^\infty e^{-\lambda(k+r)} g(\lambda) d\lambda$ is Laplace transform $g(\lambda)$ and hence :

$$f(x = k) = \binom{n}{k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} L_\lambda(k+r) I_{(0,\infty)}(x) \dots \dots \dots (2)$$

Properties of Binomial Mixture Distribution:

Lemma (1.3.1)

If X is distributed Binomial mixture distribution with $n \in \mathbf{Z}^+$, $p=e^{-\lambda}$, then factorial moment of order j given by :

$$\mu_j(X) = \frac{n!}{(n-j)!} \mu_\lambda(-j) \quad j = 1, 2, 3, \dots \quad (3)$$

Where $\mu_\lambda(j)$ is moment generating function for any distribution such as Erlang, poisson and weighted lindley distribution.

Proof :

According [1], [4], we can define the factorial moment of order j as:

$$\begin{aligned}
 \mu_j(X) &= E_\lambda[\mu_j(X/P = e^{-\lambda})] \quad , \\
 \mu_j(X/P = e^{-\lambda}) &= \frac{n!}{(n-j)!} (e^{-\lambda j}) \\
 \mu_j(X) &= E_\lambda \left[\frac{n!}{(n-j)!} (e^{-\lambda j}) \right] \\
 &= \frac{n!}{(n-j)!} E_\lambda(e^{-\lambda j}) = \frac{n!}{(n-j)!} \mu_\lambda(j)
 \end{aligned}$$

And $\mu_\lambda(j) = L_\lambda(j)$. So we get $\mu_j(X) = \frac{n!}{(n-j)!} L_\lambda(j)$, $j = 1, 2, ..$

Lemma (1.3.2) :

If X is distributed Binomial mixture distributions then the 1st, and , 3rd and the 4th moments about the origin are follows respectively:

$$1) E(X) = n L_\lambda(1) \dots \dots \dots (4)$$

$$2) E(X^2) = n L_\lambda(1) - n L_\lambda(2) + n^2 L_\lambda(2) \dots \dots \dots (5)$$

$$3) E(X^3) = n L_\lambda(1) - (3n - 3n^2) L_\lambda(2) + (n^3 - 3n^2 - 2n) L_\lambda(3) \dots \dots (6)$$

$$E(X^4) = (4n^2 - 7n) L_\lambda(2) + (12n - 6n^3 - 6n^3 - 12n^2) L_\lambda(3) + n L_\lambda(1) + (8n^4 - 8n^2 - 12n^3 - 6n) L_\lambda(4) \dots \dots (7)$$

Proof :

Either we can prove this lemma using the following formula in [6],

$$E(X^j) = E_\lambda[E(X^j/P = e^{-\lambda})] \quad j = 1, 2, 3, \dots$$

E_λ denotes the expectation with respect to the distribution of λ

Hence

$$1) E(X) = E_\lambda[E(X/P)] = E(nP) = nE(P)$$

Where $E(X/P)$ is expectation of Binomial distribution, but $p = e^{-\lambda}$

$$E(X) = n E(e^{-\lambda}) = n L_\lambda(1)$$

Or we can prove this lemma using Lemma (1.3.1) at $j=1$

$$\mu_1(X) = \frac{n!}{(n-1)!} \mu_\lambda(-1) = n E(e^{-\lambda}) = n L_\lambda(1)$$

$$2) E(X^2) = E_\lambda[E(X^2/P = e^{-\lambda})]$$

$$E(X^2/P = e^{-\lambda}) = n e^{-\lambda} - n e^{-2\lambda} + n^2 e^{-2\lambda}$$

$$\begin{aligned} E(X^2) &= E_\lambda(n e^{-\lambda} - n e^{-2\lambda} + n^2 e^{-2\lambda}) \\ &= n E_\lambda(e^{-\lambda}) - n E_\lambda(e^{-2\lambda}) + n^2 E_\lambda(e^{-2\lambda}) \\ &= n L_\lambda(1) - n L_\lambda(2) + n^2 L_\lambda(2) \end{aligned}$$

$$3) E(X^3) = E_\lambda[E(X^3/P = e^{-\lambda})]$$

$$E(X^3/P = e^{-\lambda}) = n e^{-\lambda} - (3n - 3n^2) e^{-2\lambda} + (n^3 - 3n^2 - 2n) e^{-3\lambda}$$

$$E(X^3) = n E_\lambda(e^{-\lambda}) - (3n - 3n^2) E_\lambda(e^{-2\lambda}) + (n^3 - 3n^2 - 2n) E_\lambda(e^{-3\lambda})$$

$$E(X^3) = n L_\lambda(1) - (3n - 3n^2) L_\lambda(2) + (n^3 - 3n^2 - 2n) L_\lambda(3)$$

$$4) E(X^4) = E_\lambda[E(X^4/P = e^{-\lambda})]$$

$$E(X^4/P = e^{-\lambda}) = (4n^2 - 7n) e^{-2\lambda} + (12n - 6n^3 - 6n^3 - 12n^2) e^{-3\lambda} + n e^{-\lambda} + (8n^4 - 8n^2 - 12n^3 - 6n) e^{-4\lambda}$$

$$E(X^4) = (4n^2 - 7n)E_\lambda(e^{-2\lambda}) + (12n - 6n^3 - 6n^3 - 12n^2)E_\lambda(e^{-1\lambda}) + nE_\lambda(e^{-\lambda}) + (8n^4 - 8n^2 - 12n^3 - 6n)E_\lambda(e^{-4\lambda})$$

$$= (4n^2 - 7n)L_\lambda(2) + (12n - 6n^3 - 6n^3 - 12n^2)L_\lambda(3) + nL_\lambda(1) + (8n^4 - 8n^2 - 12n^3 - 6n)L_\lambda(4)$$

Proposition (1.3.3)

If X is distrusted binomial mixture distribution then the variance , coefficient of variation , skweness , kurtosis respectively given by:

1) $Var(X) = nL_\lambda(1) - nL_\lambda(2) + n^2L_\lambda(2) + n^2(L_\lambda(1))^2 \dots \dots \dots (8)$

2) $CV = \frac{\sqrt{nL_\lambda(1) - nL_\lambda(2) + n^2L_\lambda(2) + n^2(L_\lambda(1))^2}}{nL_\lambda(1)} \dots \dots \dots (9)$

3) $SK = [nL_\lambda(1) - (3n - 3n^2)L_\lambda(2) + (n^3 - 3n^2 - 2n)L_\lambda(3) - 3n^2L_\lambda(1) + (3n^2 - 3n^3)L_\lambda(2)L_\lambda(1) + 2n^3(L_\lambda(1))^3] / \sigma^3 \dots \dots \dots (10)$

4) $KU = [(4n^2 - 7n)L_\lambda(2) + (12n - 6n^3 - 6n^3 - 12n^2)L_\lambda(3) + nL_\lambda(1) + (2n^4 - 8n^2 - 12n^3 - 6n)L_\lambda(4) - 4n^2(L_\lambda(1))^2 + (12n^3 + 12n^2)L_\lambda(1)L_\lambda(2) + (-4n^4 + 12n^3 + 8n^2)L_\lambda(3)(L_\lambda(1))^2 - 6n^3L_\lambda(2)(L_\lambda(1))^2 + 6n^3(L_\lambda(1))^2L_\lambda(1) - 3n^4(L_\lambda(1))^4] / \sigma^4 \dots \dots \dots (11)$

Proof :

According to Lemaa (1.3.2)

1) $Var(X) = E(X^2) - (E(X))^2$

$$= nL_\lambda(1) - nL_\lambda(2) + n^2L_\lambda(2) + n^2(L_\lambda(1))^2$$

2) $C.V = \frac{\sqrt{nL_\lambda(1) - nL_\lambda(2) + n^2L_\lambda(2) + n^2(L_\lambda(1))^2}}{nL_\lambda(1)}$

3) $SK = [E(X^3) - 3E(X^3)E(X) + 3E(X)(E(X))^2 - (E(X))^3] / \sigma^3$

$$= [nL_\lambda(1) - (3n - 3n^2)L_\lambda(2) + (n^3 - 3n^2 - 2n)L_\lambda(3) - 3n^2L_\lambda(1) + (3n^2 - 3n^3)L_\lambda(2)L_\lambda(1) + 2n^3(L_\lambda(1))^3] / \sigma^3$$

4) $KU = [E(X^4) - 4E(X^3)E(X) + 6E(X^2)(E(X))^2 - 4E(X)(E(X))^3 + (E(X))^4] / \sigma^4$

$$= [(4n^2 - 7n)L_\lambda(2) + (12n - 6n^3 - 6n^3 - 12n^2)L_\lambda(3) + nL_\lambda(1) + (2n^4 - 8n^2 - 12n^3 - 6n)L_\lambda(4) - 4n^2(L_\lambda(1))^2 + (12n^3 + 12n^2)L_\lambda(1)L_\lambda(2) + (-4n^4 +$$

$$\frac{12n^3 + 8n^2}{3n^4} \left[L_\lambda(3)(L_\lambda(1))^2 - 6n^3 L_\lambda(2)(L_\lambda(1))^2 + 6n^3(L_\lambda(1))^2 L_\lambda(1) - (L_\lambda(1))^4 \right] / \sigma^4$$

Some Binomial Mixtures Distribution

Binomial Mixture of Erlang Distribution:

According to Erlang Distribution, then we can define the p.m.f of random variable λ is given as:

$$g(\lambda; \theta, \alpha) = \frac{\lambda^{\theta-1} \alpha^\theta e^{-\alpha\lambda}}{\Gamma\theta} \quad \alpha, \theta > 0, \lambda > 0 \dots \dots \dots (12)$$

Definition(1.4.1.1) We say that a random variable X has a Binomial – Erlang distribution if admits the representation :

$$X/p \sim B(n, p = e^{-\lambda}) \quad \text{and} \quad \lambda \sim \text{Erlang}(\alpha, \theta) \quad \text{with} \quad \alpha, \theta > 0, n > 0$$

The distribution of the (B - E r) is characterized by three parameters α, θ, n

Lemma(1.4.1.2) If X has Erlang distribution with α, θ parameters , then the Laplace transformation of X as followings:

$$L_\lambda(S) = \left(\frac{\alpha}{\alpha + S} \right)^\theta = \mu_\lambda(-S) \dots \dots \dots (13)$$

Proof :

$$\begin{aligned} L_\lambda(S) &= E(e^{-\lambda S}) = \int_0^\infty e^{-\lambda S} \frac{\lambda^{\theta-1} \alpha^\theta e^{-\alpha\lambda}}{\Gamma\theta} d\lambda \\ &= \frac{(\alpha)^\theta}{(\alpha + S)^\theta} \int_0^\infty \frac{\lambda^{\theta-1} (\alpha + s)^\theta e^{-(\alpha+s)\lambda}}{\Gamma\theta} d\lambda \\ &= \left(\frac{\alpha}{\alpha + S} \right)^\theta = \mu_\lambda(-S) \end{aligned}$$

Where $\mu_\lambda(S) = E(e^{\lambda S}) = \frac{(\alpha)^\theta}{(\alpha+S)^\theta} \int_0^\infty \frac{\lambda^{\theta-1} (\alpha+s)^\theta e^{-(\alpha-s)\lambda}}{\Gamma\theta} d\lambda$

Theorem (1.4.1.3)

Let $X \sim B - \text{Erlang} (n , \alpha , \theta)$ the probability density function of X is given by :

$$f(x = k; \alpha, \theta, n) = \binom{n}{k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \left(\frac{\alpha}{\alpha + k + r} \right)^\theta I_{(0,\infty)}(x) \quad (14)$$

$n \in \mathbb{Z}^+ , \alpha , \theta > 0 , x = 0 , 1 , 2 , \dots , n$

Proof :

a) Method of moments:

Let $X / p \sim B(n, p=e^{-\lambda})$ and $\lambda \sim \text{Erlang}(\alpha, \theta)$ using (1)

$$\begin{aligned}
 f(x = k) &= \sum_{r=0}^{n-k} \frac{(-1)^{j-k} n!}{k! (j-k)! (n-j)!} E(p^j) I_{(0,\infty)}(x) \\
 &= \sum_{r=0}^{n-k} \frac{(-1)^{j-k} n!}{k! (j-k)! (n-j)!} E(e^{-\lambda j}) \\
 &= \sum_{r=0}^{n-k} \frac{(-1)^{j-k} n!}{k! (j-k)! (n-j)!} \mu_{\lambda}(-j)
 \end{aligned}$$

Where $\mu_{\lambda}(-j)$ is moment generating function of Erlang distribution, then the p.m.f of B-E (n, α, θ) is finally give as:

$$f(x; n, \theta) = \sum_{r=0}^{n-k} \frac{(-1)^{j-k} n!}{k! (j-k)! (n-j)!} \left(\frac{\alpha}{\alpha + k + r} \right)^{\theta} I_{(0,\infty)}(x)$$

Where $j=k+r$

b) Laplace transform :

By substituting the Laplace transform of Erlang distribution (13) into formula for mixing binomial, or using (2) directly:

$$f(x = k) = \binom{n}{k} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \left(\frac{\alpha}{\alpha + k + r} \right)^{\theta} I_{(0,\infty)}(x)$$

$$L_{\lambda}(s) = \left(\frac{\alpha}{\alpha + s} \right)^{\theta}, \quad s = k + r$$

The following figure are of the p.d.f of B-Erlang for different values of θ, α :

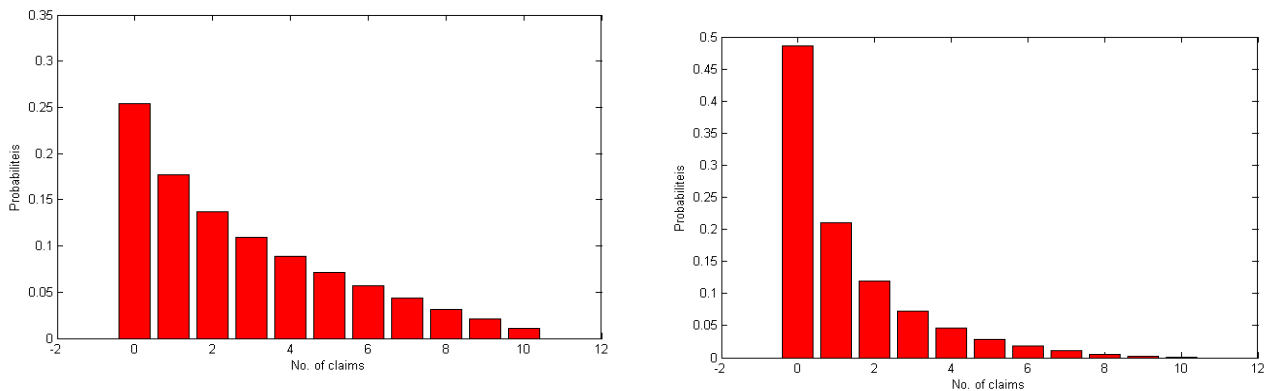
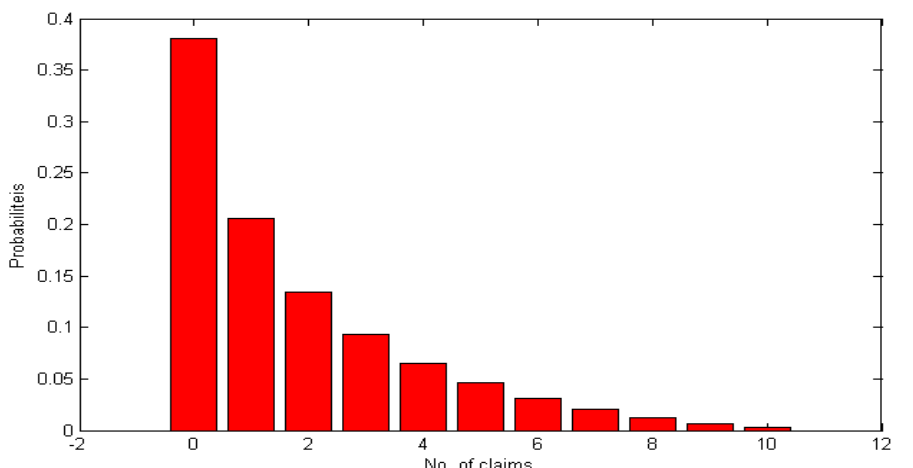


Fig.(1.1) plot of p.d.f for B-E($\theta = 1.9, \alpha = 1$)

Fig.(1.2) plot of p.d.f for B-E($\theta =3$, $\alpha=1$)



Theore

$$\mu_j(X) = \frac{n!}{(n-j)!} \left(\frac{\alpha}{\alpha+j} \right)^\theta \quad j = 1, 2, 3, \dots \quad (15)$$

proof :

According to Lemma (1.3.1), that is $\mu_j(X) = \frac{n!}{(n-j)!} \mu_\lambda(-j)$

And Lemma (1.3.2) that is $\mu_\lambda(S) = \left(\frac{\alpha}{\alpha-S} \right)^\theta$, $-s = j$

Then

$$\mu_j(X) = \frac{n!}{(n-j)!} \left(\frac{\alpha}{\alpha+j} \right)^\theta$$

Proposition 1.4.1.5 :

If X is distributed Binomial – Erlang distribution then its 1st , 2nd , 3rd , and 4th moments about the origin are as follows respectively :

1) $E(X) = n S_1 \dots\dots\dots(16)$

2) $E(X^2) = n^2 S_2 - n S_2 + n S_1 \dots\dots\dots(17)$

3) $E(X^3) = (n^3 - 3n^2 + 2n) S_3 + (3n^2 - 3n) S_2 + n S_1 \dots\dots(18)$

4) $E(X^4) = (4n^2 - 7n) S_2 + (12n - 6n^3 - 6n^3 - 12n^2) S_3 + n S_1 + (8n^4 - 8n^2 - 12n^3 - 6n) S_4 \dots\dots\dots(20)$

Where $S_1 = \left(\frac{\alpha}{\alpha+1} \right)^\theta$ $S_3 = \left(\frac{\alpha}{\alpha+3} \right)^\theta$
 $S_2 = \left(\frac{\alpha}{\alpha+2} \right)^\theta$ $S_4 = \left(\frac{\alpha}{\alpha+4} \right)^\theta$

Proof: By using factorial moments of Binomial – Erlang distribution:

$$\mu_j(X) = \frac{n!}{(n-j)!} \left(\frac{\alpha}{\alpha+j} \right)^\theta$$

And

$$L_\lambda(1) = \mu_\lambda(-1) = S_1 = \left(\frac{\alpha}{\alpha+1} \right)^\theta$$

$$L_\lambda(2) = \mu_\lambda(-2) = S_2 = \left(\frac{\alpha}{\alpha+2} \right)^\theta$$

$$L_\lambda(3) = \mu_\lambda(-3) = S_3 = \left(\frac{\alpha}{\alpha+3} \right)^\theta$$

$$L_\lambda(4) = \mu_\lambda(-4) = S_4 = \left(\frac{\alpha}{\alpha+4} \right)^\theta$$

$$1) E(X) = nL_\lambda(1) = n\mu_\lambda(-1) = n \left(\frac{\alpha}{\alpha+1} \right)^\theta = nS_1$$

$$\begin{aligned} 2) E(X^2) &= nL_\lambda(1) - nL_\lambda(2) + n^2L_\lambda(2) \\ &= n\mu_\lambda(-1) - n\mu_\lambda(-2) + n^2\mu_\lambda(-2) \\ &= nS_1 - nS_2 + n^2S_2 \end{aligned}$$

$$\begin{aligned} 3) E(X^3) &= nL_\lambda(1) - (3n - 3n^2)L_\lambda(2) + (n^3 - 3n^2 - 2n)L_\lambda(3) \\ &= nS_1 - (3n - 3n^2)S_2 + (n^3 - 3n^2 - 2n)S_3 \end{aligned}$$

$$\begin{aligned} 4) E(X^4) &= (4n^2 - 7n)L_\lambda(2) + (12n - 6n^3 - 6n^3 - 12n^2)L_\lambda(3) + \\ &\quad nL_\lambda(1) + (8n^4 - 8n^2 - 12n^3 - 6n)L_\lambda(4) \\ &= (4n^2 - 7n)S_2 + (12n - 6n^3 - 6n^3 - 12n^2)S_3 + nS_1 \\ &\quad + (8n^4 - 8n^2 - 12n^3 - 6n)S_4 \end{aligned}$$

CONCLUSIONS

We conclude that we can introduce new Binomial mixture distribution, Binomial-Erlang Distribution (B-Er) based on the transformation $p = e^{-\lambda}$ where p on the interval $[0, \infty)$ by using Moments methods and Laplace transform in mixing binomial distribution with Erlang distribution. And this new distribution is also discrete distribution for $x = 0, 1, 2, \dots, n$ with the parameters $n \in \mathbf{Z}_+$, α , $\theta > 0$.

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