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BEGINNING ARITHMETIC PROPOSITION: THE ARITHMETIC OPERATOR REQUIRES THREE ELEMENTS

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ABSTRACT: *For those not fully satisfied with abstract algebra's presentation of group and ring theory, the proposition that the arithmetic operator requires three elements is offered as a part of a new approach to algebra.*

KEYWORDS: addition, arithmetic, elements, division, identity element, multiplication, operator, set theory.

There is a beginning arithmetic. It starts with the arithmetic operator that moves in a uniform and sequential manner to order the elements of a set, and requires a minimum of three elements to produce a non-trivial result, meaning a sum that does not use zero, the identity element of arithmetic.

Specific in its execution, the arithmetic operator uses two elements in a set to compute another element in the set. Compared to a function, which is commonly expressed in the form of y = f(x) and maps a domain onto a range, or a set onto a set, an operator makes a computation using specific elements. In contrast, functions tend to use variables instead of specific elements, and algebraic expressions that involve multiple operators such as addition, multiplication, and exponents.

The term operator is also used to describe a standard function such as a partial derivative that is used to build algebraic expressions that consist of the standard function, as seen in differential equations.

Why does the arithmetic operator require three elements? Because, using either one or two elements it only performs an identity operation, which does not move within the set, and does not provide new information.

An example of an identity operation is found by adding one, the first natural number and most obvious element to use in beginning arithmetic, to the arithmetic identity element of zero or 1 + 0 = 1, which repeats the 1. Returning the non-zero element, an identity operation does not move within the set, or provide new information.

More generally, with one element, A:

A + A = A requires that A = 0, the identity element because, by definition, the identity element returns the same element. In other words, the only element that is added to itself and does not return a new element is the identity element. Alternatively, subtracting A from both sides of the equation, results in A = 0.

Or, completing the sum results in 2A = A. This requires that A = 0 since zero is the only element able to solve this equation, or the only solution. In other words, zero is the only

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element which multiplication does not change, and instead changes every other element to itself under multiplication.

Adding a second element B, either:

A + B = A, which requires that B is zero, or

A + B = B, which requires that A is zero.

With two elements, the arithmetic operator merely repeats the A or B, and does not provide new information.

But adding together two non-zero elements, the arithmetic operator creates new elements, and provides new information. For example, adding one to itself, results in a sum of two, or 1 + 1 = 2. This computation requires three elements, counting each 1 as an element, and 2 as a new element, which provides new information.

Indeed, adding one to one, and repeating the addition of one to the sum obtained, the arithmetic operator is able to generate the natural numbers in their natural sequence of 1, 2, 3, and so on.

In other words, the equation A + 1 = B can generate the set of natural numbers in their natural sequence. Starting with A = 1, results in B = 2. Moving B = 2 to A and repeating the operation, results in B = 3, and so on.

Using this simple iteration, the addition operator generates the natural numbers in their natural sequence. If the iteration uses two instead of one, it generates all the multiples of two, or the even numbers. Similar results follow using three, four, and so forth, so that this iteration forms the basis for the times tables of multiplication.

As an aside, the arithmetic operator obtains its sum as a fixed result, rather than a probability function. The sum is a specific number or element rather than a range of numbers or elements with different probabilities.

Discussion of Rings

To form a ring or a finite set where an operator reproduces all the elements of the set, going around it like a ring, as it operates over the natural numbers the addition operator requires a reverse image of the natural numbers, and the identity element of zero.

A simple example of an arithmetic ring appears in the set of {-1, 0, 1}, or more generally {-A, 0, A}. The addition of any two elements results in an element within the set and the operator is able to reproduce all the elements of the set. The elements are symmetrical, consisting of 1, its reverse image of -1, and the identity element of zero.

In general, to generate a reverse image of the natural numbers, -1 can be used in lieu of 1 in the iteration A + 1= B to result in A – 1 = B, which generates the negative integers in their natural sequence. Starting with A = -1, results in B = -2. Moving B = -2 to A and repeating the operation, results in -3, and so on. Zero, the identity element, forms a natural dividing point between the natural numbers and negative integers.

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Interestingly, rather than a set without any elements, zero often represents a state of balance or of equilibrium in a physical system. For example, zero represents a state of balance in the electric charge of an atom with equal numbers of electrons and protons. While the atom shows a net charge of zero, it possesses the ability to show a charge, letting it form chemical bonds or become a medium for the movement of electricity. Zero also represents a state of balance in a scale of weights.

Based on an arithmetic ring of the form {-A, 0, A} and the fact the arithmetic operator requires three elements to produce a non-trivial result, the proposition comes up that an arithmetic ring requires three elements since the addition operator itself requires a minimum of three elements. A simple ring appears in the set {0, 0, 0}.

The arithmetic operator includes subtraction. While subtraction first appears as the reverse of addition, it is an operator in its own right, able to operate independently over a set without requiring that it reverse a previous addition. As a general rule, addition adds to a number, making it larger. Subtraction takes away from a number, making it smaller. Subtraction is used in problems that naturally call for taking away, such as balancing a checkbook or bank account.

Although subtraction is a reverse operation of addition, it differs from addition in several important ways. First, where addition is commutative, subtraction is not commutative. In other words, where A + B = B + A, A – B does not equal B – A.

Second, where addition over the set of natural numbers results in a sum that is a natural number, subtraction over set of the natural numbers can result in elements that lie outside the set, using an element less than or equal to the element being subtracted, or an A less than or equal to B in A – B.

In other words, as subtraction operates over the natural numbers, it generates elements that lie outside the set and form part of a larger set, the integers, comprised of the natural numbers, negative integers, and the identity element of zero.

Regarding the negative integers, the first element, -1, has an interesting symmetry as a multiplier, the second type of arithmetic operator under the category of multiplication and division, with its ability to reverse the direction of 1, while having the same scalar magnitude or quantity.

In other words, multiplication by -1 changes a positive number to a negative number, with the same scalar magnitude or quantity, and changes a negative number to a positive number, with the same scalar magnitude or quantity. And -1 is a class inverter, able to change the set of natural numbers to the negative integers by multiplying the set of natural numbers by -1.

But unlike 1, -1 is not an identity element where 1 x a = a, and 1 x 1 = 1, or a x a = a. This is because -1 x a = -a, and using -1 = a, a x a = -a. However, the property of -1 as a class inverter is similar to i or the square root of -1, another class inverter that is used to describe the imaginary numbers.

As i = √-1, i x i = -1 and i is used as a multiplier in the form of ia + b to describe a complex number with a non-complex component of b, this replicates the form of -1 as a class inverter where i x a = ia, and a x a = -a.

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Regarding multiplication over the natural numbers, an argument appears where a multiplication ring requires three elements, consisting of element a, its reverse image of 1/a, and the identity element of one, or {1/a, 1, a}.

However, two other forms of a multiplication ring with three elements appear in {1, 1, 1}, since 1 x 1 = 1, and {-1, -1, 1} as follows:

-1 x -1 = 1

-1 x 1 = -1

1 x -1 = -1

This ring is similar to the ring of {1, 1, 1} since the multiplication of the two elements of -1 result in 1, which gives it essentially the same format. Thus, a multiplication ring with three elements, which does not include the identity element of 1 or -1, generally requires a reverse image of one of the elements.

Question: Is division multiplication?

As children we learn the times tables, an important tool for carrying out multiplication between numbers, and a step for the multiplication of larger numbers. Often presented in symmetrical form, the times tables are usually based on a small set of counting or natural numbers such as one to twelve, since common counting measures often use terms like a dozen eggs, and 12 is easily divisible by 2, 3, 4, and 6.

The times tables are usually comprised of an equal number of columns and entries that go down a column like a row. They start with a column for the ones, whose first entry is one times one equals one. The next entry is one times two equals two, then one times three equals three, and so on, with a last entry of one times twelve equals twelve.

The next column for the twos starts with two times one equals two. The next entry is two times two equals four, and so on, with a last entry of two times twelve equals twenty-four. The next column for the threes follows the same format, and so on, with a last column for the twelves whose first entry is twelve times one equals twelve, and whose last entry is twelve times twelve equals one hundred and forty-four.

First introduced as a fast way to add the same number a given number of times, the times tables may be used to introduce ideas of area as in the area of a square or rectangle. Later, units of physical measurement may be added, making multiplication more substantial in a physical sense than moving across a one-dimensional number line.

Soon after, we learn the reverse of multiplication called division, which moves in a reverse direction away from the multiplication product back to its individual factors. For example, where multiplication teaches that 2 x 3 = 6, or that 2 multiplied by a factor of 3 results in a product of 6, division teaches that 6 ÷ 3 = 2, or 6 divided by a divisor of 3 results in 2, meaning 2 pieces of 3.

Division involves the same three numbers as multiplication. It divides the divisor (or the multiplication factor) back into the multiplication product to obtain the number originally multiplied. Key to division is the divisor, which divides a number, amount, or element into pieces.

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Like addition and subtraction, multiplication makes larger, while division makes smaller. But multiplication and division are different from addition or subtraction because they are based on ideas of area and subdivision rather than moving across a number line, faster in their calculation process, and have a different identity element.

Moreover, multiplication and division often involve different units of measurement. In contrast, addition and subtraction are typically one-dimensional, operating over the same set or line and unit of measurement.

Like subtraction, division is an operator in its own right. It is able to both reverse the multiplication product, and be used independently of multiplication as some problems naturally call for dividing a number, amount, or object.

Similar to the differences between subtraction and addition, division is not commutative while multiplication is commutative. In other words, where a x b = b x a, a ÷ b does not equal b ÷ a.

Moreover, division over the natural numbers can result in numbers outside the set, ratios of numbers that are not integral and fractional, often called rational numbers, which form a reverse image of the natural numbers under division.

But division is not universal in its ability to operate or select elements or numbers. It has an important exception. Mathematics prohibits division by zero, the identity element for arithmetic. In contrast, addition, subtraction, and multiplication are free to operate or select elements over their respective sets of operation.

Mathematics prohibits division by zero because division requires a divisor that can divide a number, amount, or object. Since zero is not able to divide a number, object, or amount, mathematics prohibits division by zero as matter of definition.

This prohibition against division by zero reflects the fact that multiplication by zero does not make a number or element larger not even as a fraction, so that multiplication by zero does not perform a multiplication, but simply turns another number into zero.

In mathematics, issues come up in the coordination of different operations that operate over the same set and have different identity elements. One of the identity elements may be seen as more powerful than the other in the sense that it returns itself rather than the same element, similar to how multiplication by zero changes the identity of every element, including one, to itself, or zero.

In other words, since multiplication by zero changes the identity of an element to itself, and hides its identity, so division by zero is prohibited because it is unable to reverse the multiplication product to reveal the individual factors.

Multiplication by zero hides the identity of an element, turning it into zero so that no multiplication was performed. Since no multiplication was performed, division by zero is prohibited. Division is unable to reverse a multiplication product of zero, to reveal the identity of the element that was multiplied by zero.

In a sense, multiplication by zero is like a black hole, which takes away information, mapping everything into a set with a single element, zero. On the other hand, division by zero is like a supernova, the explosion of a star, a sudden, uncontrolled release of numbers or information.

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In another sense, division by zero is like a transition point, not necessarily an impossible event, but a point of instability, such as the unstable movement of a compass needle at the magnetic north pole, which wants to point in every direction.

Related to division by zero is the use of infinity as a limit, which represents an increasing number at the end of a number line that keeps going out into space. In a sense, infinity is the only number, or number concept, for which multiplication by zero does not result in zero because infinity is not defined as a specific number or element.

Thus, division by infinity is not defined since infinity is not defined as a number. Division requires a divisor that is a non-zero number, defined under a multiplication product.

Thus, while division is related to multiplication as its reverse operation, it is more sophisticated with its use of an algorithm to find factors and remainders, and is predicated upon a multiplication being able to be performed.

Enclosure

One idea of enclosure is that an operator operating over a set generates elements found within the set, or operates in a closed manner over a set. For example, used to reverse a multiplication product, the natural numbers are enclosed under division. But treating division as an independent operator, which draws from the set of natural numbers by itself, division can generate numbers outside the set, which requires ratios of natural numbers, or the rational numbers for enclosure.

Similarly, treating subtraction as the reverse of addition, the natural numbers are enclosed under subtraction. But treating subtraction as an independent operator, which draws from the set of natural numbers by itself, subtraction can generate numbers outside the set, which requires the negative integers and the identity element of zero for enclosure.

Another idea of enclosure is that an operator can use a single element or subset to generate all the elements of a set over which it operates. This implies an operator can use the same element twice. For example, adding one to itself, and repeating the operation, generates the natural numbers in their natural sequence.

Shifting this discussion to multiplication, multiplication is unable to generate the natural numbers using a single element, but using the prime numbers, which form a substantially smaller subset of the natural numbers, it can generate the natural numbers, although in a complex process.

In summary, one idea of enclosure is that an operator operating over a set of elements results in elements found within the set. A second idea of enclosure is that an operator can use a single element or subset to generate all the elements of the set over which it operates.

Other Thoughts

In division, a number is often factored into indivisible factors of prime numbers to assist in the arithmetic by canceling like terms or numbers in the numerator and denominator. This foreshadows the factoring of algebraic expressions, which allows like terms in the numerator and denominator to be canceled.

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However, as in arithmetic, algebra prohibits division by zero, using a denominator that has a term equal to zero. Otherwise, division by zero can be used to manipulate algebraic expressions to show that a number or algebraic expression is equal to any other number, one of the effects of division by zero as a sudden, uncontrolled released of numbers or information.

The integral and derivative are like multiplication and division, but applied to functions to calculate areas and rates of change. Using the limit where Δx approaches zero over a range, an integral multiplies $f(x)$ by Δx. A derivative divides the difference of $f(x + \Delta x)$ and $f(x)$ by Δx, using the limit where Δx approaches zero.

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