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An Analytical Method for Solving Wave Equations on Transmission Lines

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ABSTRACT: *Integral transform methods are very useful for solving problems in ordinary and partial differential equations. Among the integral transform methods, the Laplace transform has been applied to solve a lot of initial and boundary value problems in science and engineering. In this paper, the Laplace transform method was used for solving general wave equations on transmission lines. The model of general wave equations on transmission lines resulted into initial value hyperbolic second order partial differential equation which was transformed into ordinary differential equation by using the Laplace transform method. The method of variation of parameters and the convolution theorem of Laplace transformation was now applied to get the final solution to the problem.*

KEYWORDS: Wave equations, Laplace Transform, Lossy transmission Line, Lossless Propagation, Variation of Parameters.

INTRODUCTION

The mathematical model for a lossy transmission line contains all the primary constants or parameters of the line. These include the resistance (R), the conductance (G), the inductance (L) and the capacitance (C). Values of all these constants are specified per unit length. But for a lossless propagation, all mechanism that would cause losses to occur has negligible effect. Therefore, in the model for lossless propagation, values of resistance (R) and conductance (G) that can make losses to occur are set to zero. In this paper, the Laplace transform method was applied to solve the general wave equations on lossy and lossless transmission lines. This Laplace transform method owes its present form to a symbolic method developed by Oliver Heaviside. It provides powerful tools in numerous fields of science, engineering and technology where the knowledge of the system's transfer function is essential, Stroud and Dexter (2003). A lot of research work had been carried out on electric power transmission lines. Paul and Andie (2010) worked on characterization of losses on lossy transmission lines, Youssef and Hackum (1989) looked at a new transmission planning model, Oke and Bamigbola (2013) worked on the minimization of losses on electric power transmission lines and Abddullah et al. (2010) looked at transmission loss minimization and power installation cost using evolutionary computation for the improvement of voltage stability while Oke (2012) considered the mathematical model for the determination of voltage and current on lossy power transmission lines. There had been little or no work on the application of Laplace transform method to the solution of general wave equations on transmission lines, hence the need for this research work.

MATERIALS AND METHODS

We shall consider an infinitesimal piece of telegraph wire as an electrical circuit, which

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consists of resistance$R\Delta x$, capacitance $C\Delta x$, inductance $L\Delta x$ and conductance $G\Delta x$. If $i(x,t)$ is the current through the wire and $v(x,t)$ is the voltage at position *x* and time t while the voltage across the resistor is [redacted], and that across the coil is $itL\Delta x$ then the equivalence circuit of transmission line is as shown in figure 1 below.

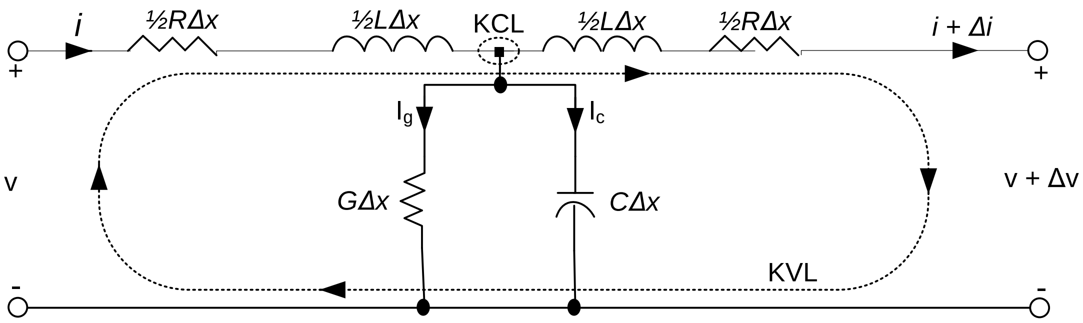

Figure 1: Equivalent Circuit of a Transmission Line

Applying the Kirchhoff's voltage law and Kirchhoff's current law in the symmetrical network of figure 1 and simplifying as appropriate, we have

$$\frac{\partial v}{\partial x} = -\left[Ri + L\frac{\partial i}{\partial t}\right] \qquad (1)$$

and

$$\boldsymbol{\frac{\partial i}{\partial x} = -\left[Gv + C\frac{\partial v}{\partial t}\right]} \qquad \mathbf{(2)}$$

Equations (1) and (2) above describe the evolution of current and voltage on a lossy transmission line, Hayt and Buck (2006) and Oke (2012).

Differentiating (1) with respect to x and (2) with respect to t and simplifying the result, we have

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + LG\frac{\partial v}{\partial t} + R\left[Gv + C\frac{\partial v}{\partial t}\right] \qquad (3)$$

Differentiating (1) with respect to t and (2) with respect to x and simplifying the result, we have

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} + G\left[Ri + L\frac{\partial i}{\partial t}\right] \qquad (4)$$

Equations (3) and (4) are hyperbolic partial differential equations which represent the general wave equations for a lossy transmission line, Hayt and Buck (2006) and Oke (2012).

Dividing equation (4) by LC, we have

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$$\frac{\partial^2 i}{\partial t^2} + \left(\frac{G}{C} + \frac{R}{L}\right)\frac{\partial i}{\partial t} + \left(\frac{G}{C}.\frac{R}{L}\right)i = \frac{1}{CL}\frac{\partial^2 i}{\partial x^2} \quad (5)$$

Let $\lambda = \frac{G}{C}$, $\beta = \frac{R}{L}$, $\phi = \frac{1}{CL}$, so that equation (5) now becomes

$$\frac{\partial^2 i}{\partial t^2} + (\lambda + \beta)\frac{\partial i}{\partial t} + \left((\lambda)(\beta)\right)i = \phi\frac{\partial^2 i}{\partial x^2} \quad (6)$$

For a lossless propagation, all the power at the input end eventually reaches the output end. This implies that power is not deviated as the wave travels down the transmission line. Therefore, all mechanisms that would cause losses to occur have negligible effect and are thereby set to zero. In our model, lossless propagation would occur when the resistance (R) and the conductance (G) are set to zero and we have

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \quad (7)$$

This implies that $\lambda = \beta = 0$ and we therefore have

$$\frac{\partial^2 i}{\partial t^2} = \phi\frac{\partial^2 i}{\partial x^2} \quad (8)$$

where $\phi = \frac{1}{CL}$

Equations (6) and (8) will now be solved with respect to the initial conditions

$i(x,0) = f(x)$, $i_t(x,0) = g(x)$

where i is the current through the conductor, $f(x)$ is the initial current and $g(x)$ is the initial speed of the current.

The Laplace transform of the partial derivatives $\frac{\partial i(x,t)}{\partial t}$ and $\frac{\partial^2 i(x,t)}{\partial t^2}$ which follows analogously from the Laplace transform of the derivatives of function of one variable are given respectively as

$$L\left[\frac{\partial i(x,t)}{\partial t}\right] = sI(x,s) - i(x,0) \quad (9)$$

and

$$L\left[\frac{\partial^2 i(x,t)}{\partial t^2}\right] = s^2 I(x,s) - si(x,0) - i_t(x,0) \quad (10)$$

As we are transforming with respect to t, we further suppose that it is legitimate to interchange differentiation and integration in the process of finding the Laplace transform of $\frac{\partial^2 i(x,t)}{\partial t^2}$, Zill and Cullen (2005). We therefore have

$$L\left[\frac{\partial^2 i(x,t)}{\partial t^2}\right] = \int_0^\infty e^{-st}\left[\frac{\partial^2 i(x,t)}{\partial t^2}\right]dt$$

$$= \int_0^\infty \frac{\partial^2}{\partial t^2}\left[e^{-st} i(x,t)\right]dt$$

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$$= \frac{d^2}{dt^2}\int_0^\infty [e^{-st} i(x,t)]dt \tag{11}$$

$\frac{d^2}{dt^2} L[i(x,t)] = \frac{d^2 I(x,s)}{dt^2}$, Dass and Verma (2011) and Riley et al. (2002). The Laplace transform of all these partial derivatives will be used extensively in this paper.

RESULTS AND DISCUSSIONS

There are several methods of solving second order partial differential equation; these include the method of separation of variables, change of variable, Fourier transform method, Laplace transform method, to name a few. The Laplace transform method is applied in this paper because the model is an initial-value problem and the initial conditions are nonhomogeneous.

Therefore, taking the Laplace transform of equation (6) with respect t and substituting the initial conditions, we have

$$s^2 I(x,s) - sf(x) - g(x) + (\lambda+\beta)sI(x,s) - (\lambda+\beta)f(x)$$

$$+((\lambda)(\beta))I(x,s) = \phi\frac{d^2 I(x,s)}{dx^2} \tag{12}$$

That is

$$\phi\frac{d^2 I(x,s)}{dx^2} - [s^2 + (\lambda+\beta)s + ((\lambda)(\beta))]I(x,s) = -[g(x) + (s+\lambda+\beta)f(x)] \tag{13}$$

which can be re-written as

$$\frac{d^2 I(x,s)}{dx^2} - b^2 I(x,s) = cy(x) \tag{14}$$

where

$$b^2 = \frac{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}{\phi}$$

and $cy(x) = -\frac{[g(x)+(s+\lambda+\beta)f(x)]}{\phi}$

The general solution of equation (14) is

$$I(x,s) = I^c(x,s) + I^p(x,s) \tag{15}$$

where $I^c(x,s)$ is the complementary function and $I^p(x,s)$ is the particular solution. Solving the associated homogeneous differential equation for (14), we have the complementary function as

$$I^c(x,s) = k_1 I^{c_1}(x,s) + k_2 I^{c_2}(x,s) = k_1 e^{bx} + k_2 e^{-bx} \tag{16}$$

Using the method of variation of parameters, Zill and Cullen (2005) and Kreyszig (1987). We seek a particular solution of the form

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$$I^p(x,s) = U_1(x)I^{c_1}(x,s) + U_2(x)I^{c_2}(x,s) \quad (17)$$

where $I^{c_1}(x,s) = e^{bx}$, $I^{c_2}(x,s) = e^{-bx}$,

$U_1(x) = \int \frac{w_1}{w} dx$ and $U_2(x) = \int \frac{w_2}{w} dx$.

$$w = \begin{vmatrix} e^{bx} & e^{-bx} \\ be^{bx} & -be^{-bx} \end{vmatrix} \quad (18)$$

$$w_1 = \begin{vmatrix} 0 & e^{-bx} \\ cy(x) & -be^{-bx} \end{vmatrix} \quad (19)$$

$$w_2 = \begin{vmatrix} e^{bx} & 0 \\ be^{bx} & -cy(x) \end{vmatrix} \quad (20)$$

We can easily see from (18), (19) and (20) that $w = -2b$, $w_1 = -cy(x)e^{-bx}$ and

$w_2 = cy(x)e^{bx}$.

Therefore $U_1(x) = \int \frac{cy(x)e^{-bx}}{2b} dx$ and $U_2(x) = -\int \frac{cy(x)e^{bx}}{2b} dx$.

Substituting the values of $U_1(x)$, $U_2(x)$, $I^{c_1}(x,s)$ and $I^{c_2}(x,s)$ in (17), we have the particular solution as

$$I^p(x,s) = \frac{e^{bx}}{2b} \int cy(x)e^{-bx} dx - \frac{e^{-bx}}{2b} \int cy(x)e^{bx} dx \quad (21)$$

The general solution to (14) is therefore

$$I(x,s) = k_1 e^{bx} + k_2 e^{-bx} + \frac{e^{bx}}{2b} \int cy(x)e^{-bx} dx - \frac{e^{-bx}}{2b} \int cy(x)e^{bx} dx \quad (22)$$

Substituting the values of b and $cy(x)$ in (22), we have

$$I(x,s) = k_1 e^{\frac{x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}} + k_2 e^{\frac{-x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}}$$

$$- \frac{e^{\frac{x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}}}{2\sqrt{\phi[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}} \int [(g(x) + (s+\lambda+\beta)f(x))e^{\frac{-x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}}]dx$$

$$+ \frac{e^{\frac{-x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}}}{2\sqrt{\phi[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}} \int [(g(x) + (s+\lambda+\beta)f(x))e^{\frac{x\sqrt{[s^2+(\lambda+\beta)s+(\lambda)(\beta)]}}{\sqrt{\phi}}}]dx \quad (23)$$

Simplifying the equation by taking $\lambda = \beta$, we have

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$$I(x,s)=k_1e^{\frac{x(s+\beta)}{\sqrt{\phi}}}+k_2e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}$$
$$-\frac{e^{\frac{x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}\int\left[g(x)e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}\right]dx-\frac{(s+2\beta)e^{\frac{x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}\int\left[f(x)e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}\right]dx$$
$$+\frac{e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}\int\left[g(x)e^{\frac{x(s+\beta)}{\sqrt{\phi}}}\right]dx+\frac{(s+2\beta)e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}\int\left[f(x)e^{\frac{x(s+\beta)}{\sqrt{\phi}}}\right] \tag{24}$$

Suppose that $g(x)$ and $f(x)$ are polynomials of degree n, then equation (24) becomes

$$I(x,s)=k_1e^{\frac{x(s+\beta)}{\sqrt{\phi}}}+k_2e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}$$
$$-\frac{e^{\frac{x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}[\sum_{i=0}^{n}(-1)^i g^i(x)\left[\int e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}dx\right]^{i+1}$$
$$-\frac{(s+2\beta)e^{\frac{x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}[\sum_{i=0}^{n}(-1)^i f^i(x)\left[\int e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}dx\right]^{i+1}$$
$$+\frac{e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}[\sum_{i=0}^{n}(-1)^i g^i(x)\left[\int e^{\frac{x(s+\beta)}{\sqrt{\phi}}}dx\right]^{i+1}$$
$$+\frac{(s+2\beta)e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}}{2(s+\beta)\sqrt{\phi}}[\sum_{i=0}^{n}(-1)^i f^i(x)\left[\int e^{\frac{x(s+\beta)}{\sqrt{\phi}}}dx\right]^{i+1} \tag{25}$$

The final solution of equation (6) together with the initial conditions will now be

$$i(x,t)=k_1e^{\frac{x\beta}{\sqrt{\phi}}}\delta\left(t+\frac{x}{\sqrt{\phi}}\right)+k_2e^{\frac{-x\beta}{\sqrt{\phi}}}\delta\left(t-\frac{x}{\sqrt{\phi}}\right)$$
$$-\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i g^i(x)e^{\frac{x\beta}{\sqrt{\phi}}}L^{-1}[\frac{e^{\frac{xs}{\sqrt{\phi}}}}{(s+\beta)}\int\left[e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}\right]dx]^{i+1}$$
$$-\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i f^i(x)e^{\frac{x\beta}{\sqrt{\phi}}}L^{-1}[\frac{(s+2\beta)e^{\frac{xs}{\sqrt{\phi}}}}{(s+\beta)}\int\left[e^{\frac{-x(s+\beta)}{\sqrt{\phi}}}\right]dx]^{i+1}$$
$$+\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i g^i(x)e^{\frac{-x\beta}{\sqrt{\phi}}}L^{-1}[\frac{e^{\frac{-xs}{\sqrt{\phi}}}}{(s+\beta)}\int\left[e^{\frac{x(s+\beta)}{\sqrt{\phi}}}\right]dx]^{i+1}$$
$$+\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i f^i(x)e^{\frac{-x\beta}{\sqrt{\phi}}}L^{-1}[\frac{(s+2\beta)e^{\frac{-xs}{\sqrt{\phi}}}}{(s+\beta)}\int\left[e^{\frac{x(s+\beta)}{\sqrt{\phi}}}\right]dx]^{i+1} \tag{26}$$

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where $\delta(t)$ is the Dirac delta function.

For a lossless propagation, where power is not dissipated as the wave travels down the transmission line, we take $\lambda = \beta = 0$ and apply the convolution theorem of Laplace transformation to get the final result as

$$i(x,t) = k_1\delta\left(t+\frac{x}{\sqrt{\phi}}\right) + k_2\delta\left(t-\frac{x}{\sqrt{\phi}}\right)$$

$$-\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i g^i(x)[\int_0^t \frac{\delta(\tau-\frac{x}{\sqrt{\phi}})}{\tau^{i+1}}.\mu[t-\left(\tau+\frac{x}{\sqrt{\phi}}\right)]d\tau]$$

$$-\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i f^i(x)[\int_0^t \frac{\delta(\tau-\frac{x}{\sqrt{\phi}})}{\tau^{i+1}}.\delta[t-\left(\tau+\frac{x}{\sqrt{\phi}}\right)]d\tau]$$

$$+\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i g^i(x)[\int_0^t \frac{\delta(\tau+\frac{x}{\sqrt{\phi}})}{\tau^{i+1}}.\mu[t-\left(\tau-\frac{x}{\sqrt{\phi}}\right)]d\tau]$$

$$+\frac{1}{2\sqrt{\phi}}\sum_{i=0}^{n}(-1)^i f^i(x)[\int_0^t \frac{\delta\left(\tau+\frac{x}{\sqrt{\phi}}\right)}{\tau^{i+1}}.\delta[t-\left(\tau-\frac{x}{\sqrt{\phi}}\right)]d\tau] \qquad (27)$$

where $\delta(t)$ is the Dirac delta function and $\mu(t)$ represents the Heaviside step function.

CONCLUSION

The Laplace transform method has been applied in this paper to solve the general wave equations on lossy and lossless transmission lines. The mathematical model for a lossy transmission line contains all the primary constants of the line. But for a lossless propagation, values of resistance and conductance were set to zero because they are part of the mechanism that would cause losses to occur in the system. The method of variation of parameters and the convolution theorem of Laplace transformation was then used to get the final solution to the problem.

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