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The same-order type A_5 characterization

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ABSTRACT: *This study examines the same-order type of G, suppose G be a finite group that α(G)=*$\{S_t | t \in \pi_e(G)\}$ *and define nse(G) or α(G) is the same-order type whose* S_t *is the number of elements of order t that* $t \in \pi_e(G)$ *and* $\pi_e(G)$ *is the set of elements order of G. If* $|G| = n$ *in G, then we say that G is* α_n*-group. Shin in* [1] *showed that every* α_2*-group is nilpotent and every* α_3*-group is solvable. In addition, the structure of the such group and proved if G be an* α_n*-group then* $|\pi(G)| \leq n$*, and conjectures that if the researcher examines the same-order type of the nonabelian simple group. This study proves that if G is a simple nonabelian group, the same-order type of G has four elements if and only if G is isomorphism with* A_5*. This study proves that for any nonabelian simple group G, we have* $S_p \neq S_q$ *for odd prime divisors p and q of order of G.*

KEYWORDS: simple group, order of elements, characterization, type of order

INTRODUCTION

One of the notable matters in recent years is the issue of distinguishing groups by nse, which is one of One of the the simple finite groups. The purpose of studying this investigation is the recognition of necessary parts which should be mentioned in the text. The first section is an introduction to these finite groups.

Let P be algebraic, it is distinguishable if any group in the category P of groups by the characteristic U in the category G. Whenever any group in category U has property P isomorphism with group G.

The set of orders of the finite groups showed by $\pi_e(G)$. Identification of groups by $\pi_e(G)$ is a broad and attractive topic. For any element like n of $\pi_e(G)$, the number of members that have order n of that group showed by S_n. In this case, the set of all these numbers is called nse of that group. It has been proven, the identifiability of some finite groups has been determined by nse [2,3]. This study shows the same group by using the same-order type for the alternating group A_5.

Actually, in the simple nonabelian group G which the same-order type of that has four-membered, then there is only A_5 which $G\cong A_5$.

This research studies the identification of simple family groups of line groups. The groups and graphs that are noticed are $C_{p,p}$-group and Suzuki groups Sz(q), prime graphs, and Ln(q). Also, we guess that for every nonabelian simple group G, exist odd prime divisors p and q of $|G|$ that $S_p \neq S_q$.

DATA ANALYSIS AND RESULTS

Characterization of group A_5 was done by same-order type. It will be defined π(n) as the set of all prime divisors n for any integer n. π(G) is defined as a set of prime numbers p so that G contains a member of order P. Now, for the finite group G, the sign $\pi_e(G)$, is the full order set of the members G. Also, we consider symbol S_t for number of members of order t in G that t∈ $\pi_e(G)$.

In the alternating group of S_3 and dihedral group D_{10}, π(G) is as below:

$$1. \pi(S_3) = \{1,2,3\}$$

$$2. D_{10} = \{e, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$$

$$\pi(D_{10}) = \{1,2,5\}$$

and $\pi_e(G)$ for alternating permutation group A_4 and dihedral group D_8 is as follows:

$$\pi_e(D_8) = \{1,2,4\} \quad , \quad \pi_e(A_4) = \{1,2,3\}$$

For example, in D_{12}, S_t is as follows:

$S_1=1$, $S_2=7$, $S_3=2$, $S_6=2$

Also, for A_4:

$S_1=1$, $S_2=3$, $S_3=8$

Specify the same-order type of group G with the symbol α(G) or nse(G). This is equal to set of $\{S_t | t \in \pi_e(G)\}$. On the other hand, the equivalence relation ~ on G is defined by

$$|g|= |h| \Leftrightarrow g \sim h$$

Then it can be seen that the set of sizes of equivalence classes is the same as α(G). For instance in D_{12} and D_8, α(G) is as follows:

$$\alpha(D_{12}) = \{S_t \mid t\in \pi_e(D_{12})\}=\{1,2,7\}$$

$$\alpha(D_8) = \{S_t \mid t \in \pi_e(D_8)\} = \{1,2,5\}.$$

The researchers also use X_n to shows the set of members of order n. Actually, S_n is the cardinal of the set S_n.

G is an α_n-group if $|\alpha(G)| = n$, clearly the only α_1-group is 1 & Z_2.

Also Z_5 is α_2-group; because $|\alpha(Z_5)| = |\{1,5\}| = 2$ and D_8 is α_3-group; because $|\alpha(D_8)| = |\{1,2,5\}| = 3$.

Shen in [3] showed that every α_2-group is nilpotent and every α_3-group is solvable. In addition, he checked the structure of such groups and conjectured that if G is an α_n-group, then $|\pi(G)| \leq n$.

For example, since in Z_p: $|\pi(Z_p)| = 2$, so Z_p is an α_2-group and nilpotent. Also in S_3, $|\pi(S_3)| = 3$ so S_3 is an α_3-group and solvable.

Lemma1. If S_k is finite, then $\phi(k)|S_k$. ($\phi(k)$ *is Eulerian function*).

Proof: if G hasn't any member of order k, Then the theorem is clear because zero is a multiple of every integer. Now assuming that a is a member of G and |a|=k, on the other hand, subgroup generated by a, has $\phi(k)$ members of order k. if all of the members of order k be in this subgroup, then $\phi(k) = S_k$ and the verdict is true. Otherwise, there is b∈G that b ∉ <a> and |b| =k. but <a> and <b> don't have a common member of order k, because if c ∈ <a> ∩ <b> and |c| = k, then <c> = <a>. Similarly, <c>=<b> then <b> = <a>, Which contradicts with assumption.

b ∉ <a>, so all cyclic subgroup of order k, have $\phi(k)$ members of order k and don't have a common member and this gives the result.

Lemma 2. Let G be a finite group and $n \in \pi_e(G)$ in this case, $n \mid \sum_{d|n} S_d$.

Proof: according to Lagrange theorem $n \mid |G|$ and since $n \mid f(n)$ (see in [3,4])

So $n \mid f(n) = \sum_{d|n} S_d$.

Remark: if $m \mid n$ then $\alpha(C_m) \subseteq \alpha(C_n)$ that C_m and C_n are cyclic groups of order m and n, respectively.

Proof: since $m \mid n$ and C_m and C_n are cyclic groups, so $(C_m) \leq (C_n)$.

On the other hand, there is a natural number h which n = mh.

Suppose $x \in \alpha(C_m)$, that $x = S_t$. we prove $x \in \alpha(C_n)$.

For every $g^h \in C_m$, we prove $x \in \alpha(C_n)$.

For every $g^h \in C_m$, we have:

$$O(g^h) = \frac{m}{(m,h)} = t \quad , \quad t \in N.$$

Also $g^{hd} \in C_n$ that

$$O(g^{hd}) = \frac{n}{(hd,n)} = \frac{nm}{(hd,md)} = t$$

So it follows

$$x = S_t \in \alpha(Cm).$$

$$x = S_t = |\{t | O(g^h) = t;\ g^h \in C_m\}| = |\{t | O(g^{hd}) = t;\ g^{hd} \in C_n\}|$$

$$\Rightarrow x \in \alpha(C_n).$$

Lemma 3. For every finite group G, $S_n = l\,\phi(n)$ which l is the the number of cyclic subgroup of order n.

Proof: we know if K be a cyclic group of order n. Then K has $\phi(K)$ members of order n. Let N and K be two cyclic subgroups of order n. If $x \in K \cap N$ be a member of order n, then because

$$<x> \leq N. <x> \leq K, \qquad |<x>| = n$$

So $<x> = K = N$.

Lemma 4. (see [5]) If G be a nonabelian simple $C_{p,p} - grop$, then G is isomorphism to the one of the following groups:

1. A_5, A_6, L_3(q);
2. L_2(q) where q is a Fermat prime or a prime power of 2;
3. Sz(q), where q is an odd prime power of 2.

Lemma 5: (see [6]) If $G \neq A_{10}$ *and* $\Gamma(G)$ is connected and G is a finite simple group, then there is three prime members r, s, t ∈ π(G) such that $\{rs, st, tr\} \cap \pi_e(G) = \emptyset$.

Lemma 6: Let G be a nonabelian finite simple group. In this case, if there is a prime number p that $S_{2p} = 0$, then the centralizer of any element of order 2 is an α_2-group.

Proof: Let $x \in G$ and $O(x) = 2^l$. In this case, we prove the centralizer of any elements of order 2 is an α_2-group, that means $|C_G(x)| = 2^l$.

We prove by contradiction if $|C_G(x)| \neq 2^{l^p}$ then $p|\, |C_G(x)|$ and p $\neq$ 2, so $\exists$ y $\in$ $C_G(x)$ that O(y)=p. Now, because O(xy)= $\frac{O(x)O(y)}{(O(x),O(y))}$ = $2^1 p \Rightarrow S_{2p} \neq 0$ which is a contradiction, so the centralizer of any element of order 2 is an α_2-group.

Lemma 7: (see [3]) Let G be a simple group. In this case, if there is a number p such that $p|\,|G|$ and p be an odd number, then $\Gamma(G)$ is connected.

Lemma 8: Let G be a finite simple nonabelian group. In this case $S_p \neq S_q$ where p and q are odd prime divisors.

Proof: Put π(G) = {2, $p_1, p_2, \dots, p_{t,}$}. Suppose on the contrary, that

$$S_{p_1} = S_{p_2} = \cdots = S_{p_t} = n.$$

In this case, since $p_i|\, p_i p_j$ and
$p_i p_j | f(p_i p_j\,)$ = 1+ $S_{p_i} + S_{p_j}$+ $S_{p_i p_j}$ so
$p_i|$1+ $S_{p_i} + S_{p_j}$+ $S_{p_i p_j}$= 1+ 2n + $S_{p_i p_j}$and also $p_i|$1+ $S_{p_i} = 1 + n$ so we have

$$\begin{cases} p_i|2 + 2S_{p_i} \\ p_i|1 + 2n + S_{p_i p_j} \end{cases} \rightarrow p_i|\, S_{p_i p_j} - 1$$

And $S_{p_i p_j} = k \in \{0, n\}$, for both distinct indexes i, j ∈ {1, 2, …, t}.

so we have the following two moods:

State 1: for all i ≤ n, if $S_{p_i p_j}$= 0, then according to the lemma 6, the centralizer of any element of order 2 is an α_2-group; that means G is a $C_{2,2} - group$ so according to lemma 4, it's enough to consider the groups G = A_6, G = $L_3(4)$. But for this group it's easy to see (using the computational group theory system GAP) that $S_p \neq S_q$for any two prime divisors p and q of |G|.

State 2: if there exists i∊{1, 2, …, n} such that $S_{2p} \neq 0$, then the prime graph of G is connected. According to the lemma 5, there exist three primes r, s, t ∊ π(G) such that

{rs, st , tr}∩ $\pi_e(G)$= ∅ which is a contradiction (for every i, j that $S_{p_i p_j} = k \neq 0$). Note that if G = A_{10}, then it is easy to see (using GAP) that there exist two odd prime divisors p and q of the │G│ such that $S_p \neq S_q$.

Lemma 9: if G be a simple group, then $S_2 \neq 1$.

Proof: state 1: if Z(G) = G, then G is abelian and according to simplicity G, it should be G ≅ Z_P.

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In this case, there are two moods for Z_P:

Case 1: if p = 2, then there is a member of order 2 that $S_2 = 1$.

Case 2: if p ≠ 2, it doesn't have a member of order 2 and $S_2 = \emptyset$. So $S_2 \neq 1$ and this is not true for Z_1. So $S_2 = 1$.

State 2: if Z(G) ≠ G, then we prove by contradiction if assume that $S_2 = 1$, in this case, $\exists\, e \neq x \in G$, so that $x^2 = 1$ and there is only member x that $x^2 = 1$, now for every g $\in G$ we show gxg^{-1} is of the order 2.
$(gxg^{-1})^2 = gxg^{-1}gxg^{-1} = gx^2g^{-1}$=g (1) g^{-1}= (1).

then since x is the only member of order 2, so for every g $\in G$, we have

$$gxg^{-1} = x \Rightarrow gx = xg$$

so x $\in Z(G)$. And since x ≠ e, thus Z(G) ≠ 1. It means 1 ≠ Z(G) $\ntrianglelefteq G$, then G is not simple.

Lemma 10: let G be a finite nonabelian simple group, then │G│has odd prime divisors p and q such that $\{1, S_2, S_p, S_q\} \subseteq \alpha(G)$.

Every nonabelian finite simple group is an α_n-group with n ≥ 4. (note that $S_2 \neq 1$, since otherwise the center of G would be nontrivial).

Lemma 11: $\varphi(n)$ is always even.

Proof: suppose n = ${p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_r}^{\alpha_r}$ that p_i are distinct prime numbers and $\alpha_i \in Z$ and also $\alpha_i \geq 1$. Firstly for every i we show $\phi({p_i}^{\alpha_i})$ is even. If $p_i = 2$ then the verdict is true. Because $\phi({p_i}^{\alpha_i}) = {p_i}^{\alpha_i} - {p_i}^{\alpha_{i-1}}$ thus, the difference of both even numbers lead to $\phi({p_i}^{\alpha_i})$ being even. Now if $p_i \neq 2$ then p_i is an odd number. So according to the relationship $\phi({p_i}^{\alpha_i}) = {p_i}^{\alpha_i} - {p_i}^{\alpha_{i-1}}$, again, the difference of both odd numbers become even. Therefore, $\phi({p_i}^{\alpha_i})$ is even, for every i, thus $\phi(n) = \phi(\prod_{i=1}^{r} {p_i}^{\alpha_i}) = \prod_{i=1}^{r} \phi({p_i}^{\alpha_i})$ that means

$\varphi(n) = \varphi({p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_r}^{\alpha_r}) = \varphi(p_1)\, \varphi(p_2) \ldots \varphi(p_r)$, because $\phi({p_i}^{\alpha_i})$ is even, as a result, their multiplication is even.

Theorem: let G be a simple nonabelian group with same-order type having four members, in this case, G ≅ A_5.

Proof: since G is a simple, so $S_2 \geq 1$, because otherwise, the center G has a member of order 2 which is a contradiction. Suppose $S_2 = m$. On the other hand, due to the simplicity of G and Burnside theorem, Π(G) ≥ 3 and according to the previous research [8] we chose the set

{2, p, q}⊆ Π(G) that $S_2 = S_p = S_q$ and Π(G) = {2, p, q}. Now according to the [7], simple nonabelian group G have exactly three prime divisors are classified into the following groups:

$$L_2(q), \quad q \in \{5, 7, 8, 9, 17\}$$

$$L_3(3), \quad U_3(3), \quad L_4(4)$$

It can be proved by using GAP software that the set of the same-order type, all of these are

α_n-group by 5 ≤ n except A_5.

CONCLUSION

According to the previous papers, if G be a simple nonabelian group, which is the set of same-order type should have four elements then will have G ≅ A_5.

If G be a simple nonabelian group, then for every odd prime divisor q and p of |G| is the result $S_p \neq S_q$.

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